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Dualities and vertex operator algebras of affine type

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Abstract

We notice that for any positive integer k, the set of $(1,2)$-specialized characters of level k standard $A_1^{(1)}$-modules is the same as the set of rescaled graded dimensions of the subspaces of level $2k+1$ standard $A_2^{(2)}$-modules that are vacuum spaces for the action of the principal Heisenberg subalgebra of $A_2^{(2)}$. We conjecture the existence of a semisimple category induced by the "equal level" representations of some algebraic structure which would naturally explain this duality-like property, and we study potential such structures in the context of generalized vertex operator algebras.
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Keywords: Affine Lie algebras; Generalized vertex operator algebras; Twisted modules

1. Introduction

In this paper we discuss a duality-like property for the rank two affine Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$, whose representation theories are to a certain extent prototypical for the representation theory of all the untwisted respectively twisted affine Lie algebras. These two algebras have therefore been intensely studied during the past three decades (cf., e.g., [1–5]).

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By abuse of notations we let Λ_0 and Λ_1 denote the fundamental weights for both algebra $A_1^{(1)}$ and algebra $A_2^{(2)}$. Let also $F_{(s_1,s_2)}$ be the (s_1,s_2)-specialization homomorphism (see Section 2.1 for notations). We start by noticing that for any positive integer k the number of level k standard $A_1^{(1)}$-modules is the same as the number of level $2k+1$ standard $A_2^{(2)}$-modules, namely $k+1$. The central theme of the paper is then the following 1–1 correspondence between the set of level k standard $A_1^{(1)}$-modules $\mathcal{O}(k) := \{L_{k_0,k}(A_1^{(1)}) \mid k_0 \in \{0,1,\ldots,k\}\}$ and the set $\{L_{k_0,2k+1}(A_2^{(2)}) \mid k_0 \in \{0,1,\ldots,k\}\}$ of level $2k+1$ standard $A_2^{(2)}$-modules:

$$\begin{aligned} &F_{(1,2)}\big(\mathrm{e}^{-(k_0\Lambda_0+(k-k_0)\Lambda_1)}\,\mathrm{ch}\,L_{k_0,k}\big(A_1^{(1)}\big)\big) \\ &\quad = P(q)^{-1}F_{(1,1)}\big(\mathrm{e}^{-(k_0\Lambda_0+(2k+1-2k_0)\Lambda_1)}\,\mathrm{ch}\,L_{k_0,2k+1}\big(A_2^{(2)}\big)\big), \end{aligned} \tag{1.1}$$

where $k_0 \in \{0,1,\ldots,k\}$ and $P(q) = \prod_{n\equiv\pm1 \bmod 6}(1-q^n)^{-1} \in \mathbf{Z}[\![q]\!]$.

The identities in (1.1) are obtained in Section 2.2 by means of the Lepowsky–Wakimoto product formulas for suitable specializations of the Weyl–Kac character formula for standard $A_1^{(1)}$- and $A_2^{(2)}$-modules, respectively. These product formulas are consequences of the techniques developed in [2,6,7], which were based mainly on the fact that the generalized Cartan matrices (GCMs) of $A_1^{(1)}$ and $A_2^{(2)}$ are conjugate. However, there is reason to believe that the identities in (1.1) are in fact consequences of a representation-theoretical phenomenon that lies deeper than initially indicated by the above-mentioned conjugation property and product formulas. Indeed, the correspondences in (1.1) may actually hint at a new kind of duality between appropriately defined module categories for $A_1^{(1)}$ and $A_2^{(2)}$, such as the category generated by the standard $A_1^{(1)}$-modules of level k and the category generated by the standard $A_2^{(2)}$-modules of level $2k+1$. These categories may be equivalently defined as the categories of level k, respectively level $2k+1$, integrable modules from the BGG category $\mathcal{O}$ for $A_1^{(1)}$ and $A_2^{(2)}$, respectively (cf. [8]). The objects in the category generated by the standard modules of level l for an untwisted affine Lie algebra $\hat{\mathfrak{g}}$ are finite direct sums of standard $\hat{\mathfrak{g}}$-modules of level l. This category is particularly important from the viewpoint of WZNW models in conformal field theory and related mathematics (see, e.g., [9–11]).

As we note at the end at Section 2.2, purely Lie-algebraic considerations alone do not seem sufficient for finding satisfactory representation-theoretical explanations for the correspondences in (1.1). We therefore propose a different yet natural approach, namely by using the representation theory of (generalized) vertex operator algebras ((G)VOAs).

Let σ be the principal automorphism of $\mathfrak{sl}(3,\mathbf{C})$ (see [3]) and denote by $\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$ the principal realization of $A_2^{(2)}$ viewed as a $\frac{1}{6}\mathbf{Z}$-graded affine Kac–Moody Lie algebra. Then the twisted affinization $\hat{\mathfrak{s}}[\sigma]$ of the principal Cartan subalgebra (CSA) $\mathfrak{s}$ of $\mathfrak{sl}(3,\mathbf{C})$ is the principal Heisenberg subalgebra of $\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$, whose representation theory explains in a natural way the presence of the factor $P(q)$ in (1.1). Indeed, the graded dimension of the twisted Fock space representation at level $2k+1$ of $\hat{\mathfrak{s}}[\sigma]$ is exactly $P(q^{1/6})$. Therefore, if $\Omega_{k_0,2k+1}$ denotes the vacuum subspace of $L_{k_0,2k+1}(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma])$ for the action of $\hat{\mathfrak{s}}[\sigma]$ and $\mathcal{O}_2(k) := \{\Omega_{k_0,2k+1} \mid k_0 \in \{0,1,\ldots,k\}\}$, we get a bijection $\phi_k : \mathcal{O}_1(k) \to \mathcal{O}_2(k)$ such that $\phi_k(L_{k_0,k}(A_1^{(1)})) = \Omega_{k_0,2k+1}$ and (1.1) may be rewritten as

$$F_{(1,2)}\bigl(\mathrm{e}^{-(k_0\Lambda_0+(k-k_0)\Lambda_1)}\,\mathrm{ch}\,L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr)\big|_{q\to q^{1/6}} = \dim_*\phi_k\bigl(L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr),\quad k_0 \in \{0,1,\ldots,k\}, \tag{1.2}$$

where $\dim_*$ denotes the graded dimension (cf. Theorem 2.2.1). From the viewpoint of representation theory, it is natural to ask whether $\mathcal{O}_1(k)$ and $\mathcal{O}_2(k)$ are the same set of (inequivalent) simple graded objects in a semisimple category induced by the "equal level" representations of some algebraic structure (Problem 1). A weaker version of this question would be to ask whether there exists an algebraic structure for which the spaces $L_{k_0,k}(A_1^{(1)})$ and $\Omega_{k_0,2k+1}$ are isomorphic irreducible graded modules for all $k_0 \in \{0,1,\ldots,k\}$ (Problem 2). We conjecture that the answer to both these questions is affirmative, and we study various possibilities for such a structure.

Let ν be the $(1,2)$-specialization automorphism of $\mathfrak{sl}(2,\mathbf{C})$. It is known that $V_k(A_1^{(1)}) := L(k\Lambda_0; A_1^{(1)})$ is a ν-rational VOA and that $L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$, $k_0 \in \{0,1,\ldots,k\}$, are all its simple ν-twisted modules (cf. Theorem 3.1.5). Although there are several ways of interpreting appropriate modifications of the left-hand side of (1.1), the most natural one appears to be that of the q-trace $\chi^{\nu}_{k_0,k}(q)$ of the simple ν-twisted $V_k(A_1^{(1)})$-module $L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$ (Theorem 3.2.1).

In order to get similar interpretations of the right-hand side of (1.1), we first construct a certain VOA $\Omega^0_{2k+1} \subset L((2k+1)\Lambda_0; A_2^{(1)})$. This is done by means of the coset (or commutant) construction associated to the irreducible Fock space representation $M(2k+1) \subset V_{2k+1}(A_2^{(1)}) := L((2k+1)\Lambda_0; A_2^{(1)})$ of the Heisenberg algebra $\tilde{\mathfrak{s}}' \subset A_2^{(1)}$. The VOA Ω^0_{2k+1} is actually a subspace of the vacuum space $\Omega_{2k+1} \subset V_{2k+1}(A_2^{(1)})$ for the action of $\tilde{\mathfrak{s}}'$, and the vacuum spaces $\Omega_{k_0,2k+1}$ defined above are σ-twisted Ω^0_{2k+1}-modules. When modified appropriately, the right-hand side of (1.1) becomes the q-trace $f_{k_0,2k+1}(q)$ of the σ-twisted Ω^0_{2k+1}-module $\Omega_{k_0,2k+1}$ (Theorem 3.2.5). Identity (1.2) may then be written as $\chi^{\nu}_{k_0,k}(q^{1/2}) = f_{k_0,2k+1}(q)$, so that in fact the equality is achieved only up to the transformation $q \to q^{1/2}$. In addition to that, both the ranks of the VOAs $V_k(A_1^{(1)})$ and Ω^0_{2k+1} and the orders of the automorphisms $\nu \in \mathrm{Aut}(V_k(A_1^{(1)}))$ and $\sigma \in \mathrm{Aut}(\Omega^0_{2k+1})$ differ by the same factor 2. By using the recently developed permutation orbifold theory [12], we can arrange to remove these differences. We endow $V_k(A_1^{(1)})^{\otimes 2}$ with a suitable VOA structure such that $V_k(A_1^{(1)})^{\otimes 2}$ is τ-rational and the level k standard $A_1^{(1)}$-modules are all its simple τ-twisted modules, where τ is a certain order 6 automorphism of $V_k(A_1^{(1)})^{\otimes 2}$. Then the

q-trace $\chi^{\tau}_{k_0,k}(q)$ of the irreducible τ-twisted $V_k(A_1^{(1)})^{\otimes 2}$-module $L_{k_0,k}(A_1^{(1)})$ coincides with $f_{k_0,2k+1}(q)$ and $\operatorname{rank} V_k(A_1^{(1)})^{\otimes 2} = \operatorname{rank} \Omega^0_{2k+1}$ (Theorem 3.2.7). Obviously, if Ω^0_{2k+1} were to fulfill the requirements of Problem 1 then it would have to satisfy a rationality-like condition. The fact that Ω^0_{2k+1} is not actually rational (cf. Remark 3.2.4(i)) would therefore suggest that Ω^0_{2k+1} may be more appropriate for a solution to Problem 2 rather than Problem 1. However, this does not necessarily rule out Ω^0_{2k+1} as a potential solution to Problem 1 as a VOA V may be σ-rational for some $\sigma \in \operatorname{Aut}(V)$ without being rational itself (cf. [13]).

Several questions arise naturally at this stage. One of these is whether $\Omega_{k_0,2k+1}$, $k_0 = 0, 1, \ldots, k$, are irreducible σ-twisted Ω^0_{2k+1}-modules. Although the results of [14] seem to indicate that this is actually true, for our purposes one would still have to deal with the question whether Ω^0_{2k+1} and $V_k(A_1^{(1)})^{\otimes 2}$ are related by some VOA homomorphism. As we point out in Appendix A (for the case $k = 1$), these VOAs are definitely not isomorphic. Besides, the fact that $V_k(A_1^{(1)})^{\otimes 2}$ contains weight one vectors while Ω^0_{2k+1} does not is also rather inconvenient in this context. We choose therefore to embed Ω^0_{2k+1} into a larger structure, namely a simple GVOA in the sense of [9] which we denote by Ω^A_{2k+1} (Theorem 4.4.3). This is done in Section 4. In addition to the fact that it certainly contains weight one vectors, we can also prove that Ω^A_{2k+1} acts irreducibly on the spaces $\Omega_{k_0,2k+1}$ without altering the q-traces $f_{k_0,2k+1}(q)$ (Theorem 4.4.8). As we explain at the end of Section 4.4, the GVOA Ω^A_{2k+1} appears to be a more appropriate object than the VOA Ω^0_{2k+1} for further investigating Problem 1. Actually, the techniques that we use in Section 4 seem in fact flexible enough to allow the construction of structures even larger than Ω^A_{2k+1} that still satisfy the same properties (cf. Remark 4.4.9). Moreover, these techniques are easily adapted to a more general situation, and they may therefore be useful for an axiomatic study of the (yet-to-be-defined) notion of "twisted module" for a GVOA. The appropriate axiomatic setting once developed, the spaces $\Omega_{k_0,2k+1}$ should provide natural examples of simple "twisted modules" for the GVOA Ω^A_{2k+1}.

The construction of Ω^A_{2k+1} and its action on $\Omega_{k_0,2k+1}$ are based on the theory of relative vertex operators and the $\mathcal{Z}$-algebra theory developed in [3,9,15]. It would be interesting to see whether structures and techniques such as simple current extensions of VOAs and intertwining operator algebras—introduced in [9,11,15–18]—provide additional tools for a continuation of the present work. Finally, one should also investigate whether similar duality-like properties hold for affine Lie algebras of higher ranks.

This paper is organized as follows: in Section 2.1 we recall some basic facts about affine Lie algebras; in Section 2.2 we prove (1.2) and formulate the problems that we study in Sections 3 and 4. In Section 3.1 we review the results on GVOAs and their modules that we need for Section 3.2, where we construct the VOAs $V_k(A_1^{(1)})$, Ω^0_{2k+1}, and $V_k(A_1^{(1)})^{\otimes 2}$ and reinterpret (1.2) in terms of the representation theory of these VOAs. In Sections 4.1–4.3 we change the setting by adapting the theories developed in [9,15] to our particular case. In Section 4.4 we use this new setting to embed Ω^0_{2k+1} into the GVOA Ω^A_{2k+1} and to prove that the latter acts irreducibly on the spaces $\Omega_{k_0,2k+1}$ without altering their q-traces.

Throughout this paper we shall use $\mathbf{N}$ for the nonnegative integers, $\mathbf{Z}_+$ for the positive integers, $\mathbf{Q}_{\geqslant 0}$ for the nonnegative rational numbers, and $\mathbf{C}^\times$ for the nonzero complex numbers.

2. A duality-like property for rank two affine Lie algebras

We first introduce a few notations and review some standard material on affine Lie algebras and their highest weight modules (further details may be found in, e.g., [6,8]). We then concentrate on the rank two affine Lie algebras and formulate the problems that we study in Sections 3 and 4.

2.1. Preliminaries and notations

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra. Fix a CSA $\mathfrak{t}$ of $\mathfrak{g}$ and let μ be an automorphism of $\mathfrak{g}$ of order r $(= 1, 2,$ or $3)$ induced by an automorphism of order r of the Dynkin diagram of $\mathfrak{g}$ with respect to $\mathfrak{t}$. Let ε be a primitive rth root of unity and denote by $\mathfrak{g}_{[i]}$ the ε^i-eigenspace of μ in $\mathfrak{g}$, $i \in \mathbf{Z}_r$. Then the fixed set $\mathfrak{g}_{[0]}$ is a simple subalgebra of $\mathfrak{g}$, the space $\mathfrak{b} := \mathfrak{g}_{[0]} \cap \mathfrak{t}$ is a CSA of $\mathfrak{g}_{[0]}$, and the $\mathfrak{g}_{[0]}$-modules $\mathfrak{g}_{[1]}$ and $\mathfrak{g}_{[-1]}$ are irreducible and contragradient. Set $l = \operatorname{rank} \mathfrak{g}_{[0]}$, and let $\{\beta_1, \ldots, \beta_l\} \subset \mathfrak{b}^*$ be a root basis of $\mathfrak{g}_{[0]}$ and $\{E_j, F_j, H_j \mid j \in \{1, \ldots, l\}\}$ a corresponding set of canonical generators of $\mathfrak{g}_{[0]}$. Let $\beta_0 \in \mathfrak{b}^*$ be the lowest weight of the $\mathfrak{g}_{[0]}$-module $\mathfrak{g}_{[1]}$, and let E_0 and F_0 be a lowest weight vector of the $\mathfrak{g}_{[0]}$-module $\mathfrak{g}_{[1]}$ respectively a highest weight vector of the $\mathfrak{g}_{[0]}$-module $\mathfrak{g}_{[-1]}$. We assume that E_0 and F_0 are normalized so that $[H_0, E_0] = 2E_0$, where $H_0 = [E_0, F_0]$. For $i, j \in \{0, 1, \ldots, l\}$ define $a_{ij} = \beta_j(H_i)$ and denote by A the matrix $(a_{ij})_{i,j=0}^{l}$. Then A is an affine GCM, and so there are positive integers $a_0, \ldots, a_l$ such that $(a_0, \ldots, a_l)A^t = 0$. Equivalently, there exist positive integers $\check{a}_0, \ldots, \check{a}_l$ such that $(\check{a}_0, \ldots, \check{a}_l)A = 0$. Both these sets of integers are assumed to be normalized so that $gcd(a_0, \ldots, a_l) = gcd(\check{a}_0, \ldots, \check{a}_l) = 1$. The integers $h := \sum_{j=0}^{l} a_j$ and $\check{h} := \sum_{j=0}^{l} \check{a}_j$ are then the Coxeter number respectively the dual Coxeter number of the matrix A. If $\mathfrak{g}$ is of type X_N $(X = A, B, \ldots, G$ and $N \geqslant 1)$, then A will be denoted by $X_N^{(r)}$. We shall use the Dynkin diagrams of the affine GCMs as listed in [19], that is, with the vertex corresponding to the 0th index always occurring at the left end of the diagram. The a_j's are then the numerical labels next to the vertices of each diagram. In particular, a_0 is always 1, while $\check{a}_0$ is 1 unless A is of type $A_{2l}^{(2)}$ $(l \geqslant 1)$, in which case $\check{a}_0 = 2$.

Let $\mathbf{s} = (s_0, s_1, \ldots, s_l)$ be a sequence of nonnegative relatively prime integers, and set $T = r \sum_{j=0}^{l} s_j a_j$. If η is a primitive Tth root of unity, the conditions

$$\nu(H_j) = H_j, \qquad \nu(E_j) = \eta^{s_j} E_j, \qquad j \in \{0, 1, \ldots, l\}, \tag{2.1.1}$$

define a Tth-order automorphism ν of $\mathfrak{g}$, the so-called $\mathbf{s}$-automorphism. The $(1,0,\ldots,0)$-automorphism is just the original diagram automorphism μ. For $j\in\mathbf{Z}_T$, let $\mathfrak{g}_{(j)}$ be the η^j-eigenspace of ν in $\mathfrak{g}$ (j denotes here both an integer between 0 and $T-1$ and its residue class modulo T). The $\mathbf{Z}_T$-gradation $\mathfrak{g}=\coprod_{j\in\mathbf{Z}_T}\mathfrak{g}_{(j)}$ is accordingly called the $\mathbf{s}$-gradation. Let $\langle\cdot,\cdot\rangle$ be a nondegenerate symmetric $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$. Being a multiple of the Killing form, $\langle\cdot,\cdot\rangle$ is also ν-invariant and remains nonsingular on the CSA $\mathfrak{b}$ of $\mathfrak{g}_{[0]}$. We may therefore identify $\mathfrak{b}$ with $\mathfrak{b}^*$ by means of the restricted form. Furthermore, we may assume that $\langle\cdot,\cdot\rangle$ is normalized so that $\langle\beta_0,\beta_0\rangle=2\check{a}_0/r$, which then implies that $\check{a}_j=r\langle\beta_j,\beta_j\rangle a_j/2$ for $j=0,1,\ldots,l$ (cf. [19, Proposition 1.1]). This normalization of the form $\langle\cdot,\cdot\rangle$ amounts to the condition that $\langle\alpha,\alpha\rangle=2$ whenever $\alpha\in\mathfrak{t}^*$ is a long root of $\mathfrak{g}$, in which case the Killing form equals $2\check{h}\langle\cdot,\cdot\rangle$. Define the Lie algebras

$$\hat{\mathfrak{g}}[\nu]=\bigoplus_{j=0}^{T-1}\mathfrak{g}_{(j)}\otimes t^{j/T}\mathbf{C}\big[t,t^{-1}\big]\oplus\mathbf{C}c,\qquad\tilde{\mathfrak{g}}[\nu]=\hat{\mathfrak{g}}[\nu]\rtimes\mathbf{C}d\tag{2.1.2}$$

by the conditions

$$\begin{aligned}&c\text{ central},\qquad c\neq 0,\qquad\big[d,a\otimes t^m\big]=ma\otimes t^m,\\&\big[a\otimes t^m,b\otimes t^n\big]=[a,b]\otimes t^{m+n}+m\delta_{m+n,0}\langle a,b\rangle c,\end{aligned}\tag{2.1.3}$$

for $m,n\in\frac{1}{T}\mathbf{Z}$, $a\in\mathfrak{g}_{(mT\bmod T)}$, $b\in\mathfrak{g}_{(nT\bmod T)}$, where $p\bmod T$ denotes the residue class of an integer p modulo T. For $a\in\mathfrak{g}_{(j)}$, $n\in\mathbf{Z}$, we shall frequently use $a(n+j/T)$ and $\mathfrak{g}(n+j/T)$ to denote $a\otimes t^{n+j/T}$ and $\mathfrak{g}_{(j)}\otimes t^{n+j/T}$, respectively, and we often identify $\mathfrak{g}_{(0)}(0)$ with $\mathfrak{g}(0)$. The space $\mathfrak{h}:=\mathfrak{b}\oplus\mathbf{C}c$ (respectively $\mathfrak{h}^e:=\mathfrak{h}\rtimes\mathbf{C}d$) is a CSA of $\hat{\mathfrak{g}}[\nu]$ (respectively $\tilde{\mathfrak{g}}[\nu]$). Let $\delta\in\mathfrak{h}^{e^*}$ be such that $\delta|_{\mathfrak{h}}=0$, $\delta(d)=1$, and define $\alpha_j\in\mathfrak{h}^{e^*}$ by $\alpha_j|_{\mathfrak{b}}=\beta_j$, $\alpha_j(c)=0$, $\alpha_j(d)=s_jT^{-1}$ if $j=1,\ldots,l$, and $\alpha_0=r^{-1}\delta-\sum_{j=1}^{l}a_j\alpha_j$, so that, in particular, $\alpha_0(d)=s_0T^{-1}$. For $j\in\{0,1,\ldots,l\}$ define also

$$\begin{aligned}&e_j=E_j\otimes t^{s_j/T},\qquad f_j=F_j\otimes t^{-s_j/T},\\&h_j=H_j+\frac{2s_j}{T\langle\beta_j,\beta_j\rangle}c.\end{aligned}\tag{2.1.4}$$

Then $\{e_j,f_j,h_j,d\mid j\in\{0,1,\ldots,l\}\}$ is a system of canonical generators of $\tilde{\mathfrak{g}}[\nu]$, viewed as the ν-twisted affine Kac–Moody Lie algebra of rank $l+1$ associated to the GCM A, in what is called the $\mathbf{s}$-realization of this algebra. Note that the canonical central element c equals $\sum_{j=0}^{l}\check{a}_jh_j$, and let $\tilde{\mathfrak{g}}[\nu]_i:=\{a\in\tilde{\mathfrak{g}}[\nu]\mid[d,a]=ia\}$, $i\in\frac{1}{T}\mathbf{Z}$. The corresponding $\frac{1}{T}\mathbf{Z}$-gradation $\tilde{\mathfrak{g}}[\nu]=\coprod_{i\in\frac{1}{T}\mathbf{Z}}\tilde{\mathfrak{g}}[\nu]_i$ is then called the $\mathbf{s}$-gradation of $\tilde{\mathfrak{g}}[\nu]$.

Remark 2.1.1. (i) Let us temporarily denote by $\{e_j^\nu,f_j^\nu,h_j^\nu,d^\nu\mid j\in\{0,1,\ldots,l\}\}$ the canonical generators of $\tilde{\mathfrak{g}}[\nu]$ defined in (2.1.4). One can show that there are uniquely determined scalars $x_1, \dots, x_l$ such that the affine Lie algebras $\tilde{\mathfrak{g}}[\mu]$ and $\tilde{\mathfrak{g}}[\nu]$ are isomorphic (but of course not graded isomorphic) by the Lie algebra map defined by $c \mapsto c$, $d^\mu \mapsto d^\nu + \sum_{k=1}^{l} x_k H_k$, $e_j^\mu \mapsto e_j^\nu$, $f_j^\mu \mapsto f_j^\nu$, $h_j^\mu \mapsto h_j^\nu$, $j = 0, 1, \dots, l$.

(ii) There are two particularly important realizations of an affine Lie algebra, namely the homogeneous and the principal ones, given by the choices $\mathbf{s} = (1, 0, \dots, 0)$ and $\mathbf{s} = (1, 1, \dots, 1)$, respectively.

Recall that a $\hat{\mathfrak{g}}[\nu]$-module V is said to be restricted if $\mathfrak{g}_{(j)}(n + j/T) \cdot v = 0$ for any $v \in V$ and $n \gg 0$. In particular, any highest weight module is restricted. Given a restricted $\hat{\mathfrak{g}}[\nu]$-module V and $a \in \mathfrak{g}_{(j)}$, we shall use generating functions of operators on V of the form

$$a(\nu; z) = \sum_{n \in \mathbf{Z}} a(n + j/T) z^{-n-j/T-1} \in (\operatorname{End} V)[\![z^{1/T}, z^{-1/T}]\!]. \tag{2.1.5}$$

We shall often write $a_{n+j/T}$ when we think of $a(n + j/T)$ as a coefficient of $a(\nu; z)$, and $a(\mathrm{id}_{\mathfrak{g}}; z)$ will be denoted by $a(z)$. We shall mostly consider highest weight modules with highest weight $\Lambda \in \mathfrak{h}^{e^*}$ satisfying $\Lambda(d) = 0$. The canonical central element $c \in \tilde{\mathfrak{g}}[\nu]$ acts on a highest weight module V with highest weight Λ as multiplication by the scalar $\Lambda(c)$, called the level of V. The character of V is defined as the formal infinite sum $\operatorname{ch} V = \sum_{\lambda \in \mathfrak{h}^{e*}} (\dim V_\lambda)\, \mathrm{e}^\lambda$, where $V = \coprod_{\lambda \leqslant \Lambda} V_\lambda$ is the weight decomposition. Note that $\mathrm{e}^{-\Lambda} \operatorname{ch} V \in \mathbf{Z}[\![\mathrm{e}^{-\alpha_0}, \dots, \mathrm{e}^{-\alpha_l}]\!]$ and let q be an indeterminate. Provided that $s_i > 0$, $0 \leqslant i \leqslant l$, the sequence $\mathbf{s} = (s_0, \dots, s_l)$ defines a homomorphism of power series rings

$$\begin{aligned} F_{\mathbf{s}} : \mathbf{Z}[\![\mathrm{e}^{-\alpha_0}, \dots, \mathrm{e}^{-\alpha_l}]\!] &\to \mathbf{Z}[\![q]\!] \\ \mathrm{e}^{-\alpha_i} &\mapsto F_{\mathbf{s}}(\mathrm{e}^{-\alpha_i}) = q^{s_i}, \quad i = 0, \dots, l, \end{aligned}$$

called the q-specialization of type $\mathbf{s}$. Let $j \in \mathbf{N}$ and set $V_j(\mathbf{s}) = \coprod_{\lambda:\, \deg\lambda = j} V_\lambda$, where $\deg(\Lambda - \sum_{i=0}^{l} k_i \alpha_i) := \sum_{i=0}^{l} k_i s_i$ for $(k_0, \dots, k_l) \in \mathbf{N}^{l+1}$. Then $V = \coprod_{j \in \mathbf{N}} V_j(\mathbf{s})$ is the $\mathbf{s}$-gradation of V and one has

$$F_{\mathbf{s}}(\mathrm{e}^{-\Lambda} \operatorname{ch} V) = \sum_{j \in \mathbf{N}} (\dim V_j(\mathbf{s})) q^j.$$

For $i = 0, 1, \dots, l$, let $\Lambda_i \in P_+$ be the fundamental weight determined by $\Lambda_i(h_j) = \delta_{ij}$, $\Lambda_i(d) = 0$. The standard $\tilde{\mathfrak{g}}[\nu]$-modules with highest weight Λ such that $\Lambda(d) = 0$ are then parameterized up to equivalence by arbitrary sequences $(k_0, k_1, \dots, k_l) \in \mathbf{N}^{l+1}$ satisfying $\Lambda = \sum_{i=0}^{l} k_i \Lambda_i$. We shall interchangeably use the notations $L_{k_0, \dots, k_{l-1}, k}(\tilde{\mathfrak{g}}[\nu])$, $L(\Lambda; \tilde{\mathfrak{g}}[\nu])$, and $L(\Lambda)$ (if no confusion is possible) in order to designate the level k standard $\tilde{\mathfrak{g}}[\nu]$-module with highest weight $\Lambda = \sum_{i=0}^{l} k_i \Lambda_i$. The $\tilde{\mathfrak{g}}[\nu]$-modules $L(\Lambda_i)$, $i = 0, 1, \dots, l$, are called fundamental, and the level one standard modules are called basic. Every standard module occurs in a tensor product of fundamental modules, and in fact every standard module of level k occurs in the tensor product of k basic modules (cf. [20]).

2.2. Rank two affine Lie algebras

We now apply the constructions described in Section 2.1 to the affine Lie algebras $A_1^{(1)}$ and $A_2^{(2)}$, and we deduce a series of coincidences between specialized characters of certain standard representations of these algebras (Theorem 2.2.1). We then describe two ways of investigating the existence of conceptual explanations for these coincidences (Problems 1 and 2).

Starting with $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$, $\mathbf{s} = (1, 0)$ and $\nu = \mathrm{id}_{\mathfrak{g}}$, one gets the "homogeneous picture" of the (untwisted) affine Lie algebra $\widehat{\mathfrak{sl}(2, \mathbf{C})}$, also denoted $A_1^{(1)}$, whose GCM is $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. Taking instead $\mathbf{s} = (1, 2)$ and the corresponding $(1, 2)$-automorphism ν of $\mathfrak{g}$ defined as in (2.1.1), one gets $\widehat{\mathfrak{sl}(2, \mathbf{C})}[\nu]$; that is, the $(1, 2)$-realization of $A_1^{(1)}$. We shall use the standard basis $\{e, f, h\}$ of $\mathfrak{sl}(2, \mathbf{C})$, with $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$.

Let now $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$ and denote by μ the negative transpose map of $\mathfrak{g}$. Then μ is a diagram automorphism of order 2 of $\mathfrak{g}$ (μ is the outer automorphism induced by the automorphism of the Dynkin diagram of $\mathfrak{g}$—with respect to some appropriate CSA of $\mathfrak{g}$—which permutes the two simple roots). Note that the fixed set of μ in $\mathfrak{g}$ is the 3-dimensional rank one subalgebra $\mathfrak{g}_{[0]} = \mathfrak{so}(3, \mathbf{C})$ ($\cong \mathfrak{sl}(2, \mathbf{C})$), while $\mathfrak{g}_{[1]}$ is the 5-dimensional subspace consisting of the symmetric traceless 3×3 matrices. The discussion of Section 2.1 with $\mathbf{s} = (1, 0)$ and μ as the $(1, 0)$-automorphism leads then to the homogeneous realization of the twisted affine Lie algebra $\widehat{\mathfrak{sl}(3, \mathbf{C})}[\mu]$, also denoted $A_2^{(2)}$, whose GCM is $\left(\begin{smallmatrix} 2 & -1 \\ -4 & 2 \end{smallmatrix}\right)$. Using instead $\mathbf{s} = (1, 1)$ and the principal automorphism σ of $\mathfrak{g}$ defined as in (2.1.1), one gets the algebra $\widehat{\mathfrak{sl}(3, \mathbf{C})}[\sigma]$, which is the principal realization of $A_2^{(2)}$.

We shall consider later on different realizations of the rank three untwisted affine Lie algebra $A_2^{(1)}$, obtained as in the previous section from $\mathfrak{g} = \mathfrak{sl}(3, \mathbf{C})$ and $\mu = \mathrm{id}_{\mathfrak{g}}$.

Let $k \in \mathbf{Z}_+$. Recall that the standard $A_1^{(1)}$-modules with a highest weight $\Lambda \in P_+$ such that $\Lambda(c) - k = \Lambda(d) = 0$ are parameterized up to equivalence by pairs $(k_0, k_1) \in \mathbf{N}^2$ satisfying

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1, \quad k_0 + k_1 = k. \tag{2.2.1}$$

Note that there are exactly $k + 1$ such distinct pairs. Suppose that $\Lambda \in P_+$ is of the form (2.2.1). If $k_0 + k_1 \neq 3k_1 + 1$, then the following infinite product expression for the $(1, 2)$-specialized character of $L_{k_0,k}(A_1^{(1)})$ is a simple consequence of identity (11.1.12) in [5]:

$$\begin{aligned} &F_{(1,2)}\left(\mathrm{e}^{-\Lambda} \operatorname{ch} L_{k_0,k}\left(A_1^{(1)}\right)\right) \\ &\quad = \prod_{\substack{n \not\equiv 0,\, 2(k+2),\, \pm(k_0+1), \\ \pm(2k-k_0+3),\, \pm 2(k-k_0+1) \bmod 4(k+2)}} \left(1 - q^n\right)^{-1}. \end{aligned} \tag{2.2.2}$$

If $k \equiv 1 \bmod 3$ and $k_0 + k_1 = 3k_1 + 1$ then identity (11.1.13) in [5] gives instead:

$$F_{(1,2)}\big(\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,L_{(2k+1)/3,k}\big(A_1^{(1)}\big)\big) = \prod_{n \equiv \pm\frac{2(k+2)}{3} \bmod 4(k+2)} \big(1-q^n\big) \prod_{\substack{n \not\equiv 0,\, 2(k+2),\, \pm\frac{2(k+2)}{3},\\ \pm\frac{4(k+2)}{3} \bmod 4(k+2)}} \big(1-q^n\big)^{-1}. \tag{2.2.3}$$

The identities in [5] mentioned above were derived from the Weyl–Kac character and denominator formulas [8] and Wakimoto's generalization of Lepowsky's numerator formula [6,7].

We now look at standard $A_2^{(2)}$-modules with a highest weight $\Lambda \in P_+$ such that $\Lambda(c) = 2k+1$ and $\Lambda(d) = 0$. These are parameterized up to equivalence by pairs $(k_0, k_1) \in \mathbf{N}^2$ satisfying

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1, \quad 2k_0 + k_1 = 2k+1. \tag{2.2.4}$$

Note that there are exactly $k+1$ such distinct pairs. Define

$$P(q) = \prod_{n \equiv \pm 1 \bmod 6} \big(1-q^n\big)^{-1} \in \mathbf{Z}[\![q]\!]$$

and let $\Lambda \in P_+$ be as in (2.2.4). The principally specialized character of $L(\Lambda; A_2^{(2)})$ was expressed as an infinite product in [2, Theorem 6.8] by means of the character, denominator and numerator formulas. One has:

$$F_{(1,1)}\big(\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,L_{k_0,2k+1}\big(A_2^{(2)}\big)\big) = P(q) \prod_{\substack{n \not\equiv 0,\, 2(k+2),\, \pm(k_0+1),\\ \pm(2k-k_0+3),\, \pm 2(k-k_0+1) \bmod 4(k+2)}} \big(1-q^n\big)^{-1} \tag{2.2.5}$$

if $k_0 \neq (2k+1)/3$, and

$$F_{(1,1)}\big(\mathrm{e}^{-\Lambda}\,\mathrm{ch}\,L_{(2k+1)/3,2k+1}\big(A_2^{(2)}\big)\big) = P(q) \prod_{n \equiv \pm\frac{2(k+2)}{3} \bmod 4(k+2)} \big(1-q^n\big) \prod_{\substack{n \not\equiv 0,\, 2(k+2),\, \pm\frac{2(k+2)}{3},\\ \pm\frac{4(k+2)}{3} \bmod 4(k+2)}} \big(1-q^n\big)^{-1} \tag{2.2.6}$$

if $k_0 = (2k+1)/3$ (which can occur only for $k \equiv 1 \bmod 3$).

It is now obvious that except for the factor $P(q)$, the right-hand sides of (2.2.2) and (2.2.3) coincide with those of (2.2.5) and (2.2.6), respectively. Therefore, for each positive integer k we have a 1–1 correspondence

$$\big\{L_{k_0,k}\big(A_1^{(1)}\big) \bigm| k_0 \in \{0,1,\ldots,k\}\big\} \leftrightarrow \big\{L_{k_0,2k+1}\big(A_2^{(2)}\big) \bigm| k_0 \in \{0,1,\ldots,k\}\big\} \tag{2.2.7}$$

such that for every $k_0 \in \{0,1,\ldots,k\}$

$$F_{(1,2)}\big(\mathrm{e}^{-(k_0\Lambda_0+(k-k_0)\Lambda_1)}\operatorname{ch} L_{k_0,k}\big(A_1^{(1)}\big)\big)$$
$$= P(q)^{-1}F_{(1,1)}\big(\mathrm{e}^{-(k_0\Lambda_0+(2k+1-2k_0)\Lambda_1)}\operatorname{ch} L_{k_0,2k+1}\big(A_2^{(2)}\big)\big).$$

The presence of the factor $P(q)$ in (2.2.5)–(2.2.6) may be explained by the representation theory of the principal Heisenberg subalgebra of $\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$, where σ is as before the principal automorphism of $\mathfrak{sl}(3,\mathbf{C})$. More precisely, $P(q)$ is the rescaled graded dimension of the fermionic twisted Fock space representation at level $2k+1$ of this Heisenberg algebra. Indeed, let $\tau:\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]\to\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]/\mathbf{C}c$ be the canonical Lie algebra map, and note that the principal gradation of $\widetilde{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$ is actually a $\frac{1}{6}\mathbf{Z}$-gradation. Consider the following subalgebras of $\widetilde{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$:

$$\tilde{\mathfrak{s}}[\sigma]=\tau^{-1}\big(\mathrm{Cent}_{\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]/\mathbf{C}c}\,\tau(e_0+e_1)\big)\rtimes\mathbf{C}d,\qquad \tilde{\mathfrak{s}}[\sigma]'=\big[\tilde{\mathfrak{s}}[\sigma],\tilde{\mathfrak{s}}[\sigma]\big],$$
$$\tilde{\mathfrak{s}}[\sigma]'_\pm=\mathbf{C}\text{-span}\big\{x\in\tilde{\mathfrak{s}}[\sigma]'\ \big|\ \pm\deg x>0\big\},\tag{2.2.8}$$

where e_0 and e_1 are as in (2.1.4). Then $\tilde{\mathfrak{s}}[\sigma]'=\tilde{\mathfrak{s}}[\sigma]'_-\oplus\mathbf{C}c\oplus\tilde{\mathfrak{s}}[\sigma]'_+$, and one can show that the commutator subalgebra of $\tilde{\mathfrak{s}}[\sigma]'$ is one-dimensional and coincides with $\mathbf{C}c$ (see Remark 3.2.3). Hence $\tilde{\mathfrak{s}}[\sigma]'$ is a Heisenberg Lie algebra, called the principal Heisenberg subalgebra of $\widetilde{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$ (cf. [3]). Define a $(\tilde{\mathfrak{s}}[\sigma]'_+\oplus\mathbf{C}c)$-module structure on $\mathbf{C}$ by $c\cdot 1=(2k+1)\cdot 1$, $\tilde{\mathfrak{s}}[\sigma]'_+\cdot 1=0$, $\deg 1=0$, and form the induced $\tilde{\mathfrak{s}}[\sigma]'$-module

$$M(\sigma;2k+1)=U\big(\tilde{\mathfrak{s}}[\sigma]'\big)\otimes_{U(\tilde{\mathfrak{s}}[\sigma]'_+\oplus\mathbf{C}c)}\mathbf{C},\tag{2.2.9}$$

which acquires a $\frac{1}{6}\mathbf{Z}$-gradation by letting d act as the degree operator. Then $M(\sigma;2k+1)$ is an irreducible $\tilde{\mathfrak{s}}[\sigma]'$-module which is isomorphic to $S(\tilde{\mathfrak{s}}[\sigma]'_-)$ as a $\frac{1}{6}\mathbf{Z}$-graded vector space. Recall that the vacuum space of an arbitrary $\tilde{\mathfrak{s}}[\sigma]'$-module V is the ($\frac{1}{6}\mathbf{Z}$-graded) subspace $\Omega_V=\{v\in V\mid\tilde{\mathfrak{s}}[\sigma]'_+\cdot v=0\}$. If $\Lambda\in P_+$ is as in (2.2.4), then [21, Theorem 1.7.3] implies that when viewed as a $\tilde{\mathfrak{s}}[\sigma]'$-module, $L_{k_0,2k+1}(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma])$ decomposes as

$$L_{k_0,2k+1}\big(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]\big)=M(\sigma;2k+1)\otimes\Omega_{k_0,2k+1}$$
$$=S\big(\tilde{\mathfrak{s}}[\sigma]'_-\big)\otimes\Omega_{k_0,2k+1},\tag{2.2.10}$$
$$\text{where}\quad \Omega_{k_0,2k+1}:=\Omega_{L_{k_0,2k+1}(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma])}$$

is the vacuum subspace of $L_{k_0,2k+1}(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma])$ for the action of $\tilde{\mathfrak{s}}[\sigma]'$. It is then easy to see that

$$F_{(1,1)}\big(\mathrm{e}^{-(k_0\Lambda_0+(2k+1-2k_0)\Lambda_1)}\operatorname{ch} L_{k_0,2k+1}\big(A_2^{(2)}\big)\big)\big|_{q\to q^{1/6}}$$
$$=\dim_* L_{k_0,2k+1}\big(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]\big)=\dim_* S\big(\tilde{\mathfrak{s}}[\sigma]'_-\big)\dim_*\Omega_{k_0,2k+1}$$
$$=P\big(q^{1/6}\big)\dim_*\Omega_{k_0,2k+1},\tag{2.2.11}$$

where $\dim_*$ denotes the graded dimension (cf. [21, Section 1.10]). Set

$$\begin{aligned} \mathcal{O}_1(k) &= \bigl\{ L_{k_0,k}\bigl(A_1^{(1)}\bigr) \bigm| k_0 \in \{0,1,\ldots,k\} \bigr\}, \\ \mathcal{O}_2(k) &= \bigl\{ \Omega_{k_0,2k+1} \bigm| k_0 \in \{0,1,\ldots,k\} \bigr\}. \end{aligned} \tag{2.2.12}$$

We conclude that (2.2.7) can be restated as follows.

Theorem 2.2.1. *For each positive integer k there is a bijection*

$$\begin{aligned} \phi_k : \mathcal{O}_1(k) &\to \mathcal{O}_2(k) \\ L_{k_0,k}\bigl(A_1^{(1)}\bigr) &\mapsto \Omega_{k_0,2k+1}, \quad k_0 \in \{0,1,\ldots,k\}, \end{aligned} \tag{2.2.13}$$

such that

$$F_{(1,2)}\bigl(\mathrm{e}^{-(k_0\Lambda_0+(k-k_0)\Lambda_1)} \operatorname{ch} L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr)\big|_{q\to q^{1/6}} = \dim_* \phi_k\bigl(L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr)$$

for every $k_0 \in \{0,1,\ldots,k\}$.

One is naturally led to assume that a deeper reason may lie behind the bijections ϕ_k, which in fact suggest a new kind of duality between appropriately defined module categories for $A_1^{(1)}$ and $A_2^{(2)}$. From the viewpoint of representation theory, natural approaches for finding conceptual explanations to Theorem 2.2.1 would be the following problems.

Problem 1. Are $\mathcal{O}_1(k)$ and $\mathcal{O}_2(k)$ the same set of inequivalent simple graded objects in a semisimple category induced by the "equal level" representations of some algebraic structure?

Problem 2. Does there exist an algebraic structure for which the spaces $L_{k_0,k}(A_1^{(1)})$ and $\Omega_{k_0,2k+1}$ are isomorphic simple graded modules for all $k_0 \in \{0,1,\ldots,k\}$?

The "equal level" representations in the formulation of Problem 1 are of course to be understood in a broad sense, by analogy with the (highest weight) modules of fixed level for an affine Lie algebra, the (unitary) *Vir*-modules of fixed central charge, the modules for a (rational) VOA with prescribed rank, etc. Problem 2 is obviously a weaker version of Problem 1 and we shall therefore concentrate mainly on the latter. We conjecture that $\mathcal{O}_1(k)$ and $\mathcal{O}_2(k)$ are in fact the same set of simple twisted modules for a (rational) VOA or a related structure, and we produce some evidence in this direction in Sections 3 and 4.

Remark 2.2.2. From a Lie-algebraic point of view, the natural candidates for the algebraic structures of Problems 1 and 2 would be some (appropriately infinite-dimensional) Lie algebras. For instance, one could try to construct suitable $A_1^{(1)}$-module structures for the spaces in $\mathcal{O}_2(k)$. Although there are several ways of embedding $A_1^{(1)}$ into $A_2^{(2)}$, these procedures are not likely to serve our purposes mainly because they fail to make the spaces $\Omega_{k_0,2k+1}$ become $A_1^{(1)}$-modules (see also Remark 3.2.6 for a somewhat similar point of view). Let us consider as an example the following elements of $A_2^{(2)}$:

$$e_0' = \frac{1}{6}\big[e_0, \big[\big[[e_0, e_1], e_1\big], e_1\big]\big], \qquad f_0' = \frac{1}{6}\big[f_0, \big[\big[[f_0, f_1], f_1\big], f_1\big]\big],$$
$$e_1' = e_1, \qquad f_1' = f_1, \qquad h_0' = 3h_1 + 8h_0, \qquad h_1' = h_1,$$

where $e_i, f_i, h_i, i = 0, 1$, are as in (2.1.4). Then $\{e_i', f_i', h_i' \mid i \in \{0, 1\}\}$ generates a subalgebra $\mathfrak{a}$ of $A_2^{(2)}$ which is isomorphic to the affine Lie algebra $A_1^{(1)}$ with canonical central element $c' = h_0' + h_1' = 4c$ (where c is the canonical central element of $A_2^{(2)}$), so that one may identify $\mathfrak{a}$ with the full subalgebra of $A_1^{(1)}$ of depth 4. However, the action of $\mathfrak{a}$ on $L(\Lambda; \widehat{\mathfrak{sl}(3, \mathbf{C})}[\sigma])$—$\Lambda \in P_+$ being as in (2.2.4)—does not centralize the action of the Heisenberg algebra $\tilde{\mathfrak{s}}[\sigma]'$ defined in (2.2.8), and so the corresponding vacuum space $\Omega_{L(\Lambda; \widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma])}$ is not necessarily $\mathfrak{a}$-invariant. Moreover, the principal realization of $A_2^{(2)}$ induces on $\mathfrak{a} \cong A_1^{(1)}$ the gradation of type $(5, 1)$—and not $(1, 2)$ as needed—and the level of $L(\Lambda; \widehat{\mathfrak{sl}(3, \mathbf{C})}[\sigma])$ viewed as an $\mathfrak{a}$-module increases to $\Lambda(c') = 4(2k + 1)$.

3. Standard modules for $A_1^{(1)}$ and $A_2^{(2)}$ as representations of VOAs

In this section we give a first approach to Problem 1. We start by noticing that after some appropriate modifications, expressions (2.2.2)–(2.2.3) and (2.2.5)–(2.2.6) become the q-traces of certain (twisted) modules for VOAs associated to vacuum representations of $A_1^{(1)}$ and $A_2^{(1)}$, respectively. As it was previously indicated, the ultimate goal would be to interpret both $\mathcal{O}_1(k)$ and $\mathcal{O}_2(k)$ as the same set of equivalence classes of irreducible (twisted) modules for some rational VOA or a related structure. We study various possibilities for such a structure.

3.1. GVOAs and modules

We review here some necessary background on GVOAs and their modules. For the formal calculus involved and further results we refer to [9,21–26].

Definition 1 [9]. Let $S \in \mathbf{Z}_+$ and let G be a finite abelian group endowed with a symmetric nondegenerate $\frac{1}{S}\mathbf{Z}/2\mathbf{Z}$-valued $\mathbf{Z}$-bilinear form:

$$(g, h) \in \tfrac{1}{S}\mathbf{Z}/2\mathbf{Z} \quad \text{for } g, h \in G.$$

A *generalized vertex operator algebra of level S* associated with the group G and the form $(\cdot,\cdot)$ is a vector space V with two gradations

$$V=\coprod_{n\in\frac{1}{S}\mathbf{Z}} V_n=\coprod_{g\in G} V^g \quad \text{with} \quad \mathrm{wt}(v)=n \quad \text{for } v\in V_n,$$

such that

$$V^g=\coprod_{n\in\frac{1}{S}\mathbf{Z}} V_n^g, \quad \text{where} \quad V_n^g=V_n\cap V^g \quad \text{for any } g\in G \text{ and } n\in \tfrac{1}{S}\mathbf{Z},$$

$$\dim V_n<\infty \quad \text{for all } n\in\tfrac{1}{S}\mathbf{Z}, \qquad V_n=0 \quad \text{for } n\ll 0,$$

which is equipped with a linear map

$$\begin{aligned} Y(\cdot\,,z):V &\to (\mathrm{End}\,V)[\![z^{1/S},z^{-1/S}]\!] \\ v &\mapsto Y(v,z)=\sum_{n\in\frac{1}{S}\mathbf{Z}} v_n z^{-n-1} \end{aligned}$$

and with two distinguished vectors $\mathbf{1}\in V_0^0$, $\omega\in V_2^0$, satisfying the following conditions: for any $g,h\in G$, $u,v\in V$ and $m\in\frac{1}{S}\mathbf{Z}$

$$u_m V^h\subset V^{g+h} \quad \text{if } u\in V^g, \qquad u_m v=0 \quad \text{if } m\gg 0,$$
$$Y(\mathbf{1},z)=\mathrm{id}_V, \qquad Y(v,z)\mathbf{1}\in V[\![z]\!] \quad \text{and} \quad \lim_{z\to 0} Y(v,z)\mathbf{1}=v,$$
$$Y(v,z)|_{V^h}=\sum_{n\equiv(g,h)\,\mathrm{mod}\,\mathbf{Z}} v_n z^{-n-1} \quad \text{if } v\in V^g$$

(i.e., $n+2\mathbf{Z}\equiv(g,h)\,\mathrm{mod}\,\mathbf{Z}/2\mathbf{Z}$); if $u\in V^g$ and $v\in V^h$ then the following generalized Jacobi identity holds:

$$\begin{aligned} &z_0^{-1}\left(\frac{z_1-z_2}{z_0}\right)^{(g,h)}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(u,z_1)Y(v,z_2) \\ &\quad - z_0^{-1}\left(\frac{z_2-z_1}{z_0}\right)^{(g,h)}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y(v,z_2)Y(u,z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y\big(Y(u,z_0)v,z_2\big)\left(\frac{z_1-z_0}{z_2}\right)^{-g}, \end{aligned}$$

where $\delta(z)=\sum_{n\in\mathbf{Z}} z^n$ and whenever $k\in G$ and $w\in V^k$,

$$\left(\frac{z_1-z_0}{z_2}\right)^{-g}\delta\left(\frac{z_1-z_0}{z_2}\right)\cdot w=\left(\frac{z_1-z_0}{z_2}\right)^{-(g,k)}\delta\left(\frac{z_1-z_0}{z_2}\right)w;$$

furthermore, for any $m,n\in\mathbf{Z}$ one has

$$\big[L(m),L(n)\big]=(m-n)L(m+n)+\tfrac{1}{12}\big(m^3-m\big)\delta_{m+n,0}(\mathrm{rank}\,V)\,\mathrm{id}_V,$$

where $L(n)=\omega_{n+1}$ for $n\in\mathbf{Z}$, i.e., $Y(\omega,z)=\sum_{n\in\mathbf{Z}}L(n)z^{-n-2}$, rank $V\in\mathbf{C}$, and

$$L(0)v=nv=\mathrm{wt}(v)v \quad \text{for } n\in\tfrac{1}{S}\mathbf{Z},\ v\in V_n,$$
$$Y\big(L(-1)v,z\big)=\frac{\mathrm{d}}{\mathrm{d}z}Y(v,z).$$

This completes the definition of a generalized vertex operator algebra (GVOA). The GVOA defined above is denoted by $(V,Y,\mathbf{1},\omega,S,G,(\cdot\,,\cdot))$ or simply by V if no confusion is possible.

A $\frac{1}{S}\mathbf{Z}$-*graded VOA* is a GVOA of level S associated with the group $G=\{0\}$. Following [27], we call a GVOA of this type a **Q**-*graded VOA*. A VOA is then a **Q**-graded VOA of level 1.

Definition 2 [9,27]. Let $(V,Y,\mathbf{1},\omega,S)$ be a **Q**-graded VOA. A *weak V-module* is a pair (M,Y_M), where M is a vector space and $Y_M(\cdot\,,z)$ is a linear map $V\to(\operatorname{End}M)[\![z^{1/S},z^{-1/S}]\!]$ satisfying the following: $Y_M(\mathbf{1},z)=\mathrm{id}_M$, $z^nY_M(a,z)u\in M[\![z^{1/S}]\!]$ for any $a\in V$, $u\in M$, and $n\in\frac{1}{S}\mathbf{Z}$ sufficiently large,

$$Y_M\big(L(-1)a,z\big)=\frac{\mathrm{d}}{\mathrm{d}z}Y_M(a,z)\quad \text{for } a\in V,$$

and the Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(a,z_1)Y_M(b,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(b,z_2)Y_M(a,z_1)$$
$$=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M\big(Y(a,z_0)b,z_2\big)\quad \text{for } a,b\in V.$$

A weak V-module (M,Y_M) is called a (ordinary) *V-module* if $L(0)$ $(=\operatorname{Res}_z z\times Y_M(\omega,z))$ acts semisimply on M with the decomposition into $L(0)$-eigenspaces $M=\coprod_{h\in\mathbf{C}}M_h$ such that for any $h\in\mathbf{C}$, $\dim M_h<\infty$ and $M_{h+n}=0$ for $n\in\frac{1}{S}\mathbf{Z}$ sufficiently small. A **Q**-*graded weak module* is a weak V-module (M,Y_M) together with a **Q**-gradation $M=\coprod_{n\in\mathbf{Q}}M_n$ such that $a_mM_n\subseteq M_{r+n-m-1}$ for $a\in V_r$, $r,m\in\frac{1}{S}\mathbf{Z}$, $n\in\mathbf{Q}$. An *admissible V-module* is a $\mathbf{Q}_{\geqslant 0}$-graded weak V-module. A **Q**-graded VOA V is said to be *rational* if every admissible V-module is completely reducible, i.e., a direct sum of simple admissible V-modules.

Let $(V,Y,\mathbf{1},\omega)$ be a VOA with an automorphism σ of order T. Set $V^k=\{a\in V\mid \sigma(a)=\exp(2k\pi\mathrm{i}/T)a\}$, $0\leqslant k\leqslant T-1$, so that $V=\bigoplus_{k=0}^{T-1}V^k$.

Definition 3 [12,28]. A *weak σ-twisted V-module* is a pair (M,Y_M), where M is a vector space and $Y_M(\cdot\,,z)$ is a linear map $V\to(\operatorname{End}M)[\![z^{1/T},z^{-1/T}]\!]$ given by $a\mapsto Y_M(a,z)=\sum_{n\in\mathbf{Q}}a_nz^{-n-1}$, such that for $a,b\in V$ and $u\in M$ the following hold: $a_nu=0$ if $n\gg 0$, $Y_M(\mathbf{1},z)=\mathrm{id}_M$, $Y_M(a,z)=\sum_{n\in k/T+\mathbf{Z}}a_nz^{-n-1}$ for $a\in V^k$, and the twisted Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(a,z_1)Y_M(b,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(b,z_2)Y_M(a,z_1)$$
$$=z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-k/T}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M\big(Y(a,z_0)b,z_2\big)\quad \text{for } a\in V^k, b\in V.$$

A weak σ-twisted V-module (M, Y_M) is called a (ordinary) *σ-twisted V-module* if $L(0)$ acts semisimply on M with the decomposition into $L(0)$-eigenspaces $M=\coprod_{h\in\mathbf{C}} M_h$ such that for any $h\in\mathbf{C}$, $\dim M_h<\infty$ and $M_{h+n}=0$ for $n\in\frac{1}{S}\mathbf{Z}$ sufficiently small. A *$\frac{1}{T}\mathbf{Z}$-graded weak σ-twisted V-module* is a weak σ-twisted V-module (M, Y_M) which carries a $\frac{1}{T}\mathbf{Z}$-grading $M=\coprod_{n\in\frac{1}{T}\mathbf{Z}} M_n$ such that $a_m M_n\subseteq M_{r+n-m-1}$ for $a\in V_r$, $r\in\mathbf{Z}$, $m,n\in\frac{1}{T}\mathbf{Z}$. An *admissible σ-twisted V-module* is a $\frac{1}{T}\mathbf{N}$-graded weak σ-twisted V-module. A VOA V is said to be *σ-rational* if every admissible σ-twisted V-module is completely reducible.

It was shown in [28] that if (M, Y_M) is a weak σ-twisted V-module then the component operators of $Y_M(\omega,z)$ together with id_M induce a *Vir*-representation on M with central charge $\operatorname{rank} V$ and $Y_M(L(-1)a,z)=(\mathrm{d}/\mathrm{d}z)Y_M(a,z)$ for $a\in V$. It was further shown in *loc. cit.* that a σ-rational VOA has only finitely many isomorphism classes of simple admissible σ-twisted modules and that every such module is an ordinary σ-twisted module.

The following are consequences of the above definitions (cf. [25]): if V is a VOA, $\sigma\in\operatorname{Aut}(V)$ has order T, $0\leqslant k\leqslant T-1$, $a\in V^k$, $b\in V$, and (M, Y_M) is a weak σ-twisted V-mod ule then

$$\big[Y_M(a,z_1),Y_M(b,z_2)\big]$$
$$=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\left(\frac{\partial}{\partial z_2}\right)^j z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\left(\frac{z_2}{z_1}\right)^{k/T}\right)Y_M(a_j b,z_2), \tag{3.1.1}$$

$$Y_M\big(Y(a,z_0)b,z_2\big)$$
$$=\operatorname{Res}_{z_1}\left(\frac{z_1-z_0}{z_2}\right)^{k/T}\left[z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(a,z_1)Y_M(b,z_2)\right.$$
$$\left.-z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)Y_M(b,z_2)Y_M(a,z_1)\right]. \tag{3.1.2}$$

These formulas reduce to the ordinary commutator respectively associator relations for VOAs by taking $M=V$ and $\sigma=\mathrm{id}_V$. As in the untwisted case (cf. [26, Lemma 1.2.1]) one can prove:

Proposition 3.1.1. *Let V be a VOA with an automorphism σ of order T. If M is a simple admissible σ-twisted V-module such that $M=\coprod_{n\in\frac{1}{T}\mathbf{N}} M_n$ and $M_0\neq 0$, $\dim M_0<\infty$, then there exists $h_M\in\mathbf{C}$ such that $L(0)|_{M_n}=(n+h_M)\,\mathrm{id}_{M_n}$ for all $n\in\frac{1}{T}\mathbf{N}$.*

By Proposition 3.1.1, one may write the q-trace (conformal character) of an irreducible σ-twisted V-module $M = \coprod_{n \in \frac{1}{T}\mathbf{N}} M_n$ as the following formal power series:

$$\operatorname{tr}_M q^{L(0)-\operatorname{rank} V/24} = q^{h_M - \operatorname{rank} V/24} \sum_{n=0}^{\infty} (\dim M_{n/T}) q^{n/T}. \tag{3.1.3}$$

Obviously, any admissible σ-twisted V-module with finite-dimensional $L(0)$-homogeneous subspaces has a well-defined q-trace.

Proposition 3.1.2 [28,29]. *Let $(V, Y, \mathbf{1}, \omega)$ be a VOA of rank $r \in \mathbf{C}$ and suppose that $h \in V$ is such that $L(n)h = \delta_{n,0}h$, $h_n h = \delta_{n,1}\lambda\mathbf{1}$ for $n \in \mathbf{N}$, where $\lambda \in \mathbf{C}$. Then $(V, Y, \mathbf{1}, \omega + h_{-2}\mathbf{1})$ is a vertex algebra of rank $r - 12\lambda$.*

Remark 3.1.3. The vector $\omega + h_{-2}\mathbf{1}$ becomes a new Virasoro element of V, on which it induces a $\mathbf{C}$-gradation instead of the original $\mathbf{Z}$-gradation without altering the other axioms in the definition of a VOA. We shall use Proposition 3.1.2 only when the new Virasoro vector induces a $\frac{1}{S}\mathbf{Z}$-gradation on V for some $S \in \mathbf{Z}_+$ so that the new structure becomes a $\mathbf{Q}$-graded VOA (cf. Definition 1).

We now describe briefly the VOAs associated with affine Lie algebras, which are sometimes called affine VOAs (cf. [30]). Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with the form $\langle\cdot,\cdot\rangle$ normalized as in Section 2.1, and form the associated untwisted affine Kac–Moody algebra $\tilde{\mathfrak{g}} = \hat{\mathfrak{g}} \rtimes \mathbf{C}d$ as in (2.1.2)–(2.1.3). Set $\tilde{\mathfrak{g}}_{\geqslant 0} = \bigoplus_{n \geqslant 0} \mathfrak{g}(n) \oplus \mathbf{C}c \oplus \mathbf{C}d$ and let $-\check{h} \neq l \in \mathbf{C}$. Define a $\tilde{\mathfrak{g}}_{\geqslant 0}$-module structure on $\mathbf{C}$ by $c \cdot 1 = l \cdot 1$, $d \cdot 1 = 0$, $\mathfrak{g}(n) \cdot 1 = 0$ for $n \geqslant 0$, and form the induced $\tilde{\mathfrak{g}}$-module $N(l\Lambda_0) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\geqslant 0})} \mathbf{C}$ (the so-called vacuum representation of $\tilde{\mathfrak{g}}$ at level l). Note that $N(l\Lambda_0)$ is a restricted $\tilde{\mathfrak{g}}$-module such that $N(l\Lambda_0) \cong U(\bigoplus_{n<0} \mathfrak{g}(n))$ as vector spaces, and that we may identify $\mathfrak{g}(-1) \otimes 1$ with $\mathfrak{g}$. Set $\mathbf{1} = 1 \otimes 1 \in N(l\Lambda_0)$ and define the element

$$\omega = \frac{1}{2(l + \check{h})} \sum_{j=1}^{\dim \mathfrak{g}} a^j(-1) b^j(-1)\mathbf{1} \in N(l\Lambda_0),$$

where $\{a^j \mid j \in \{1, \ldots, \dim \mathfrak{g}\}\}$ and $\{b^j \mid j \in \{1, \ldots, \dim \mathfrak{g}\}\}$ are dual bases of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$. Recall from (2.1.5) the series $a(z)$ in this case and define the map

$$\begin{aligned} Y : \mathfrak{g}(-1)\mathbf{1} &\to \big(\operatorname{End} N(l\Lambda_0)\big)[\![z, z^{-1}]\!] \\ Y\big(a(-1)\mathbf{1}, z\big) &= a(z), \quad a \in \mathfrak{g}. \end{aligned} \tag{3.1.4}$$

One can show (cf., e.g., [5, Theorem 2.6]) that Y extends uniquely to $N(l\Lambda_0)$ in such a way that $N(l\Lambda_0)$ becomes a VOA with vacuum vector $\mathbf{1}$ and Virasoro element ω such that $\mathfrak{g}(-1)\mathbf{1} = N(l\Lambda_0)_1$ (the weight one subspace of $N(l\Lambda_0)$).

Moreover, given any restricted $\hat{\mathfrak{g}}$-module M of level l, there is a canonical extension to $N(l\Lambda_0)$ of the map

$$\begin{aligned} Y_M : \mathfrak{g}(-1)\mathbf{1} &\to (\operatorname{End} M)\big[\!\big[z, z^{-1}\big]\!\big] \\ Y_M\big(a(-1)\mathbf{1}, z\big) &= a(z), \quad a \in \mathfrak{g}, \end{aligned}$$

such that (M, Y_M) becomes a weak $N(l\Lambda_0)$-module (cf. [5,9,24,29,31]). Let finally $N^1(l\Lambda_0)$ be the (unique) maximal proper $\tilde{\mathfrak{g}}$-submodule of $N(l\Lambda_0)$ and notice that we may identify the irreducible quotient $N(l\Lambda_0)/N^1(l\Lambda_0)$ with the standard $\tilde{\mathfrak{g}}$-module $L(l\Lambda_0)$. We summarize some of the results of the above-mentioned papers in the following theorem.

Theorem 3.1.4. *For each $l \neq -\check{h}$, $(N(l\Lambda_0), Y, \mathbf{1}, \omega)$ is a VOA of rank $(l \dim \mathfrak{g})/(l + \check{h})$ generated by $\mathfrak{g}(-1)\mathbf{1}$ and any restricted $\hat{\mathfrak{g}}$-module of level l is a weak $N(l\Lambda_0)$-module. Every $\tilde{\mathfrak{g}}$-submodule of $N(l\Lambda_0)$ is an ideal of $N(l\Lambda_0)$ viewed as a VOA. In particular, there exists an induced structure of simple VOA on $L(l\Lambda_0)$. If l is a positive integer, then $L(l\Lambda_0)$ is rational and the set of equivalence classes of simple $L(l\Lambda_0)$-modules is exactly the set of equivalence classes of standard $\hat{\mathfrak{g}}$-modules of level l.*

Any automorphism σ of order T of $\mathfrak{g}$ preserves the form $\langle\cdot,\cdot\rangle$ and induces a Lie algebra automorphism of $\tilde{\mathfrak{g}}$. Since the VOA $N(l\Lambda_0)$ is generated by $N(l\Lambda_0)_1 = \mathfrak{g}(-1)\mathbf{1}$, it follows from the associator formula for VOAs that σ induces a VOA automorphism of $N(l\Lambda_0)$ (cf. [29, Proposition 6.20]). It is easy to see that $N^1(l\Lambda_0)$ is invariant under the induced VOA automorphism. Hence σ induces a VOA automorphism of $L(l\Lambda_0)$ as well. We denote these induced VOA automorphisms again by σ. If M is a restricted $\hat{\mathfrak{g}}[\sigma]$-module of level l, the map

$$\begin{aligned} Y_M^\sigma : \mathfrak{g}(-1)\mathbf{1} &\to (\operatorname{End} M)\big[\!\big[z^{1/T}, z^{-1/T}\big]\!\big] \\ Y_M^\sigma\big(a(-1)\mathbf{1}, z\big) &= a(\sigma; z), \end{aligned} \tag{3.1.5}$$

for $a \in \mathfrak{g}_{(j)}$, $j = 0, \ldots, T-1$, has a unique extension to $N(l\Lambda_0)$ that makes (M, Y_M^σ) a weak σ-twisted $N(l\Lambda_0)$-module. This is a consequence of the theory of local systems of twisted vertex operators developed in [25], where the following result was obtained.

Theorem 3.1.5. *Let $l \neq -\check{h}$ be a complex number. Then any restricted $\hat{\mathfrak{g}}[\sigma]$-module of level l is a weak σ-twisted $N(l\Lambda_0)$-module. If l is a positive integer, then $L(l\Lambda_0)$ is σ-rational and the set of equivalence classes of simple σ-twisted $L(l\Lambda_0)$-modules is precisely the set of equivalence classes of standard $\hat{\mathfrak{g}}[\sigma]$-modules of level l.*

Given $a \in \mathfrak{g}_{(j)}$, $b \in \mathfrak{g}$, and a restricted $\hat{\mathfrak{g}}[\sigma]$-module M of level l, one gets from the affine Lie algebra relations (2.1.3) that

$$\begin{aligned}\big[a(\sigma;z_1),b(\sigma;z_2)\big] &= z_1^{-1}\left(\frac{z_2}{z_1}\right)^{j/T}\delta\left(\frac{z_2}{z_1}\right)[a,b](\sigma;z_2)\\ &\quad+\langle a,b\rangle\frac{\partial}{\partial z_2}\left(z_1^{-1}\delta\left(\frac{z_2}{z_1}\right)\left(\frac{z_2}{z_1}\right)^{j/T}\right)l\,\mathrm{id}_M \qquad (3.1.6)\end{aligned}$$

in $(\operatorname{End} M)[\![z_1^{1/T},z_1^{-1/T},z_2^{1/T},z_2^{-1/T}]\!]$. Following [9], we define a (noncommutative) "normal ordering" operation for $a\in\mathfrak{g}_{(j)}$, $b\in\mathfrak{g}_{(k)}$, $m\in j/T+\mathbf{Z}$, $n\in k/T+\mathbf{Z}$ by

$$\,^{\times}_{\times}a_mb_n\,^{\times}_{\times}=\begin{cases}a_mb_n, & \text{if } m<0,\\ b_na_m, & \text{if } m\geqslant 0.\end{cases}$$

Using the computational techniques of [25] one can then easily prove the following lemma.

Lemma 3.1.6. *Let M be a restricted $\hat{\mathfrak{g}}[\sigma]$-module of level l, and let $a\in\mathfrak{g}_{(k)}$, $b\in\mathfrak{g}_{(-k)}$ for some fixed $k\in\{0,\dots,T-1\}$. Then*

$$\begin{aligned}Y_M^{\sigma}\big(a(-1)b(-1)\mathbf{1},z\big) &= \sum_{s\in\mathbf{Z}}\bigg(\sum_{n\in\mathbf{Z}}\,^{\times}_{\times}a(n+k/T)b\big(s-(n+k/T)\big)\,^{\times}_{\times}\\ &\quad-\frac{k}{T}[a,b](s)-\langle a,b\rangle\delta_{s,0}\binom{\frac{k}{T}}{2}l\,\mathrm{id}_M\bigg)z^{-s-2}.\end{aligned}\tag{3.1.7}$$

In particular,

$$\begin{aligned}\operatorname{Res}_z zY_M^{\sigma}\big(a(-1)b(-1)\mathbf{1},z\big) &= \sum_{n\in\mathbf{Z}}\,^{\times}_{\times}a(n+k/T)b\big(-(n+k/T)\big)\,^{\times}_{\times}\\ &\quad-\frac{k}{T}[a,b](0)-\langle a,b\rangle\binom{\frac{k}{T}}{2}l\,\mathrm{id}_M\,. \qquad (3.1.8)\end{aligned}$$

Recall finally that given two VOAs $(V_i,Y_{V_i},\mathbf{1}_{V_i},\omega_{V_i})$, $i=1,2$, one may define the tensor product VOA $(V_1\otimes V_2,Y_\otimes,\mathbf{1}_\otimes,\omega_\otimes)$ by setting

$$\begin{aligned}&\mathbf{1}_\otimes=\mathbf{1}_{V_1}\otimes\mathbf{1}_{V_2},\qquad \omega_\otimes=\omega_{V_1}\otimes\mathbf{1}_{V_2}+\mathbf{1}_{V_1}\otimes\omega_{V_2},\\ &Y_\otimes(a_1\otimes a_2,z)=Y_{V_1}(a_1,z)\otimes Y_{V_2}(a_2,z)\quad \text{for } a_i\in V_i.\end{aligned}\tag{3.1.9}$$

The new VOA is endowed with the tensor product grading and the central charge for the Virasoro algebra relations becomes $\operatorname{rank}(V_1\otimes V_2)=\operatorname{rank}V_1+\operatorname{rank}V_2$.

3.2. Structures associated to standard modules for $A_1^{(1)}$, $A_2^{(1)}$, and $A_2^{(2)}$

We start our discussion of Problem 1 with some remarks on the objects in $\mathcal{O}_1(k)$, namely the level k standard $A_1^{(1)}$-modules. By Theorem 3.1.4, $V_k(A_1^{(1)}):=$

$L(k\Lambda_0; A_1^{(1)})$ is a rational VOA of rank $c_1(k) := 3k/(k+2)$ and the level k standard $A_1^{(1)}$-modules are precisely its simple modules. These modules have q-traces which are essentially the same as their homogeneously specialized characters when viewed as $A_1^{(1)}$-modules, but we need a setting that makes the former agree with the $(1,2)$-specialization of the latter. There are two natural and compatible ways to achieve this, as we shall now explain.

Recall from Section 2.1 the standard basis $\{e, f, h\}$ of $\mathfrak{sl}(2, \mathbf{C})$ and the form $\langle\cdot,\cdot\rangle$, and note that $\langle h, h\rangle = 2\langle e, f\rangle = 2$, $\langle h, e\rangle = \langle h, f\rangle = 0$. A Virasoro element of $V_k(A_1^{(1)})$ is then given by

$$\omega = \frac{1}{2(k+2)}\left[e(-1)f(-1)\mathbf{1} + f(-1)e(-1)\mathbf{1} + \frac{1}{2}h(-1)^2\mathbf{1}\right] \in V_k\big(A_1^{(1)}\big)_2. \tag{3.2.1}$$

Let $\widetilde{\omega} = \omega + \frac{1}{3}h(-2)\mathbf{1}$ and $Y(\widetilde{\omega}, z) = \sum_{n\in\mathbf{Z}} \widetilde{L}(n)z^{-n-2}$. Then

$$\widetilde{L}(0) = L(0) - \tfrac{1}{3}h(0), \quad \text{and} \tag{3.2.2}$$

$$\begin{aligned} &\big[\widetilde{L}(0), e(n)\big] = -\big(n + \tfrac{2}{3}\big)e(n), \qquad \big[\widetilde{L}(0), f(n)\big] = -\big(n - \tfrac{2}{3}\big)f(n), \\ &\big[\widetilde{L}(0), h(n)\big] = -nh(n) \quad \text{for } n \in \mathbf{Z}. \end{aligned} \tag{3.2.3}$$

By Proposition 3.1.2 and Remark 3.1.3, $(V_k(A_1^{(1)}), Y, \mathbf{1}, \widetilde{\omega})$ is a $\frac{1}{3}\mathbf{Z}$-graded VOA of rank $\tilde{c}_1(k) := c_1(k) - 8k/3$. Since $k \in \mathbf{Z}_+$, it follows from [32, Theorem 3.7] that the set of inequivalent simple weak $(V_k(A_1^{(1)}), Y, \mathbf{1}, \widetilde{\omega})$-modules is exactly $\mathcal{O}_1(k)$. Moreover, (3.2.3) implies that for $n \in \mathbf{N}$, all $e(n)$, $f(n+1)$, and $h(n+1)$ have negative degrees with respect to the operator $\widetilde{L}(0)$, so that by [27, Theorem 2.20] any weak (in particular, any admissible) $(V_k(A_1^{(1)}), Y, \mathbf{1}, \widetilde{\omega})$-module is completely reducible. Thus $(V_k(A_1^{(1)}), Y, \mathbf{1}, \widetilde{\omega})$ is a rational $\frac{1}{3}\mathbf{Z}$-graded VOA of rank $\tilde{c}_1(k)$ and $\mathcal{O}_1(k)$ is the complete set of its inequivalent simple modules (cf. [27, Theorem 3.14]). By analogy with (3.1.3), the q-traces of these modules are the modified graded dimensions induced by $\widetilde{L}(0)$:

$$\begin{aligned} \chi_{k_0,k}(q) &= \operatorname{tr}_{L_{k_0,k}(A_1^{(1)})} q^{\widetilde{L}(0) - \tilde{c}_1(k)/24} \\ &= q^{h_{k_0,k} - \tilde{c}_1(k)/24} \sum_{n=0}^{\infty} \big(\dim L_{k_0,k}\big(A_1^{(1)}\big)_{n/3}\big) q^{n/3}, \end{aligned} \tag{3.2.4}$$

where $h_{k_0,k} \in \mathbf{C}$ is the lowest weight of $L_{k_0,k}(A_1^{(1)})$ with respect to $\widetilde{L}(0)$. Using (3.1.8), (3.2.1), and (3.2.2) one easily gets that

$$h_{k_0,k} = \frac{3k_0^2 - 2k_0k - k^2 + 2k_0 - 2k}{12(k+2)}, \tag{3.2.5}$$

while (3.2.3) implies that

$$\sum_{n=0}^{\infty}\bigl(\dim L_{k_0,k}\bigl(A_1^{(1)}\bigr)_{n/3}\bigr)q^{n/3}=F_{(1,2)}\bigl(\mathrm{e}^{-k_0\Lambda_0-k_1\Lambda_1}\,\mathrm{ch}\,L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr)\big|_{q\to q^{1/3}},$$

which substituted in (3.2.4) gives

$$\chi_{k_0,k}(q)=F_{(1,2)}\bigl(\mathrm{e}^{-k_0\Lambda_0-k_1\Lambda_1}\,\mathrm{ch}\,L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr)\big|_{q\to q^{1/3}}\cdot q^{h_{k_0,k}-\tilde{c}_1(k)/24}. \tag{3.2.6}$$

Up to the indicated modifications, the characters (2.2.2) and (2.2.3) may therefore be interpreted as the q-traces of the simple modules for the rational $\frac{1}{3}\mathbf{Z}$-graded VOA $(V_k(A_1^{(1)}),Y,\mathbf{1},\widetilde{\omega})$.

Alternatively, one may view these modified characters as the q-traces of the simple ν-twisted modules over the ν-rational VOA $(V_k(A_1^{(1)}),Y,\mathbf{1},\omega)$, where ν is the $(1,2)$-automorphism of $\mathfrak{sl}(2,\mathbf{C})$ given by $\nu(h)=h$, $\nu(e)=\exp(4\pi\mathrm{i}/3)e$, $\nu(f)=\exp(2\pi\mathrm{i}/3)f$. Indeed, let us denote the canonical generators of $\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu]$ as in Remark 2.1.1 and notice that $h=\frac{1}{3}(h_1^{\nu}-2h_0^{\nu})$. Obviously, the degree operator d^{ν} of $\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu]$ induces a $\frac{1}{3}\mathbf{Z}$-gradation on $L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$, and one has

$$\begin{aligned}&\sum_{n=0}^{\infty}\bigl(\dim L_{k_0,k}\bigl(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu]\bigr)_{n/3}\bigr)q^{n/3}\\&\quad=F_{(1,2)}\bigl(\mathrm{e}^{-k_0\Lambda_0-k_1\Lambda_1}\,\mathrm{ch}\,L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr)\big|_{q\to q^{1/3}}.\end{aligned}\tag{3.2.7}$$

On the other hand, $\{(L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu]),Y^{\nu})\mid k_0\in\{0,1,\ldots,k\}\}$ is the complete set of simple ν-twisted modules for the ν-rational VOA $(V_k(A_1^{(1)}),Y,\mathbf{1},\omega)$ by Theorem 3.1.5. From the twisted commutator formula (3.1.1) one gets that

$$\begin{aligned}&\bigl[L(0),e(n+2/3)\bigr]=-(n+2/3)e(n+2/3),\\&\bigl[L(0),f(n+1/3)\bigr]=-(n+1/3)f(n+1/3),\end{aligned}\tag{3.2.8}$$

where $L(0)=\mathrm{Res}_z\,zY^{\nu}(\omega,z)$. By (3.2.7) and (3.2.8), $L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$ has the q-trace

$$\begin{aligned}\chi_{k_0,k}^{\nu}(q)&:=\mathrm{tr}_{L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])}\,q^{L(0)-c_1(k)/24}\\&=F_{(1,2)}\bigl(\mathrm{e}^{-k_0\Lambda_0-k_1\Lambda_1}\,\mathrm{ch}\,L_{k_0,k}\bigl(A_1^{(1)}\bigr)\bigr)\big|_{q\to q^{1/3}}\cdot q^{h_{k_0,k}^{\nu}-c_1(k)/24},\end{aligned}\tag{3.2.9}$$

where $h_{k_0,k}^{\nu}\in\mathbf{C}$ is the lowest weight of $L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$ with respect to the operator $L(0)$ of (3.2.8). The value of $h_{k_0,k}^{\nu}$ is easily obtained from (3.1.8) and (3.2.1):

$$h_{k_0,k}^{\nu}=\frac{9k_0^2-6k_0k+k^2+6k_0+2k}{36(k+2)}.\tag{3.2.10}$$

Then (3.2.5) and (3.2.10) yield

$$h_{k_0,k} - \frac{\tilde{c}_1(k)}{24} = h^{\nu}_{k_0,k} - \frac{c_1(k)}{24} = \frac{18k_0^2 - 12k_0k + 2k^2 + 12k_0 - 5k}{72(k+2)}. \tag{3.2.11}$$

It now follows from (3.2.6), (3.2.9), and (3.2.11) that

$$\chi_{k_0,k}(q) = \chi^{\nu}_{k_0,k}(q). \tag{3.2.12}$$

Summarizing, we have the following theorem.

Theorem 3.2.1. (i) $(V_k(A_1^{(1)}), Y, \mathbf{1}, \widetilde{\omega})$ *is a simple rational* $\frac{1}{3}\mathbf{Z}$*-graded VOA of rank* $\tilde{c}_1(k)$ *and the complete set of its inequivalent simple modules is* $\{(L_{k_0,k}(A_1^{(1)}), Y) \mid k_0 \in \{0, 1, \ldots, k\}\}$.

(ii) $(V_k(A_1^{(1)}), Y, \mathbf{1}, \omega)$ *is a simple* ν*-rational VOA of rank* $c_1(k)$ *and the complete set of its inequivalent simple* ν*-twisted modules is* $\{(L_{k_0,k}(\widehat{\mathfrak{sl}(2, \mathbf{C})})[\nu]), Y^{\nu}) \mid k_0 \in \{0, 1, \ldots, k\}\}$.

(iii) *The* q*-trace of* $(L_{k_0,k}(A_1^{(1)}), Y)$ *and the* q*-trace of* $(L_{k_0,k}(\widehat{\mathfrak{sl}(2, \mathbf{C})}[\nu]), Y^{\nu})$ *satisfy*

$$\begin{aligned} \chi_{k_0,k}(q) &= \chi^{\nu}_{k_0,k}(q) \\ &= F_{(1,2)}\big(\mathrm{e}^{-k_0\Lambda_0 - k_1\Lambda_1} \operatorname{ch} L_{k_0,k}\big(A_1^{(1)}\big)\big)\big|_{q \to q^{1/3}} \cdot q^{h^{\nu}_{k_0,k} - c_1(k)/24} \end{aligned}$$

for every $k_0 \in \{0, 1, \ldots, k\}$.

As we mentioned in the introduction of Section 3, an ideal answer to Problem 1 would be that $\mathcal{O}_1(k)$ and $\mathcal{O}_2(k)$ are in fact the same set of simple modules for a rational VOA-like structure such that appropriate modifications of the characters (2.2.2)–(2.2.3) are the q-traces of these modules. If true, this would almost certainly require some modular invariance properties from the functions $\widetilde{\chi}_{k_0,k}(\tau) := \chi_{k_0,k}(\mathrm{e}^{2\pi \mathrm{i}\tau})$, where $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, $\Im(\tau) > 0$. Indeed, it was proved in [26] that the linear span of the q-traces of all the simple V-modules becomes a (finite-dimensional) module for the modular group in case V is rational and C_2-finite, i.e., the space $\{a_{-2}b \mid a, b \in V\}$ has finite codimension in V. This modular invariance property has been generalized in a suitable sense to ν-rational C_2-finite VOAs in [33]. Since most of the known ν-rational VOAs—including $(V_k(A_1^{(1)}), Y, \mathbf{1}, \omega)$—are C_2-finite (cf. *loc. cit.*), Theorem 3.2.1 implies that the characters $\widetilde{\chi}_{k_0,k}$ do in fact satisfy the modular invariance property in the sense of [33]. Moreover, it can be shown that $\widetilde{\chi}_{k_0,k}$ is a modular function for some subgroup of finite index of the modular group [34].

Remark 3.2.2. The above construction of ν-twisted modules from untwisted modules follows from [25, Proposition 5.4]. A different but essentially equivalent construction was given in [17] by using the restricted dual and a certain automorphism, a procedure which in our case basically amounts to the change of Virasoro element. Since in the present situation all finite order automorphisms are inner, the previous constructions are actually a reflection of the fact that under the deformed action associated with the semisimple weight one primary vector $\lambda h = \lambda h_{-1}\mathbf{1}$, $\lambda \in \mathbf{Q}$, the simple (adjoint) module $V_k(A_1^{(1)})$ is a so-called G-simple current in physical terminology, where G is any torsion subgroup of $\mathrm{Aut}(V_k(A_1^{(1)}))$ that contains $\exp(2\pi \mathrm{i}\lambda h(0))$ (cf. [16]). Although such deformed actions also work for general modules, the simple current modules are particularly important from a physical point of view. This is mainly because they give rise to a tensor functor which acts as a permutation on the set of equivalence classes of irreducible weak modules (the associated matrix of the left multiplication of the equivalence class of a simple current module with respect to the standard basis of the Verlinde algebra is a permutation). We refer to [16–18] for an in-depth discussion of simple current (twisted) modules and various simple current extensions of VOAs.

We now concentrate on the objects in $\mathcal{O}_2(k)$, namely the spaces $\Omega_{k_0,2k+1}$ defined in (2.2.10). More specifically, we show that these spaces are twisted modules for a certain VOA lying inside $L((2k+1)\Lambda_0; A_2^{(1)})$ in such a way that their q-traces are well defined and actually equal to $\chi^{\nu}_{k_0,k}(q^{1/2})$.

Recall the setting and notations of Sections 2.1–2.2. In particular, μ is the minus transpose map of $\mathfrak{sl}(3,\mathbf{C})$, and $\mathfrak{sl}(3,\mathbf{C})_{[i]}$ is the $(-1)^i$-eigenspace of μ, $i=0,1$. Since $\mathfrak{sl}(3,\mathbf{C})_{[0]} \cong \mathfrak{sl}(2,\mathbf{C})$, we may choose a basis $\{E_1, F_1, H_1\}$ of $\mathfrak{sl}(3,\mathbf{C})_{[0]}$ such that $[H_1,E_1]=2E_1$, $[H_1,F_1]=-2F_1$, and $[E_1,F_1]=H_1$. Then $H_0 := [E_0,F_0] = -H_1/2$, where E_0 and F_0 are a lowest respectively highest weight vector of the $\mathfrak{sl}(3,\mathbf{C})_{[0]}$-module $\mathfrak{sl}(3,\mathbf{C})_{[1]}$, normalized so that $[H_0,E_0]=2E_0$. The CSA $\mathfrak{s}$ of $\mathfrak{sl}(3,\mathbf{C})$ is chosen to be the principal one; that is

$$\mathfrak{s} = \mathrm{Cent}_{\mathfrak{sl}(3,\mathbf{C})}(E_0+E_1) = \mathbf{C}\text{-span}\{E_0+E_1, 2F_0+F_1\}. \tag{3.2.13}$$

The form $\langle\cdot\,,\cdot\rangle$ being normalized as in Section 2.1, one has that

$$\begin{aligned}&\langle E_0,F_0\rangle = 1, \qquad \langle E_1,F_1\rangle = 4, \qquad \langle H_1,H_1\rangle = 8,\\ &[H_1,E_0] = -4E_0, \qquad [H_1,F_0] = 4F_0.\end{aligned} \tag{3.2.14}$$

Let $\eta = \exp(\pi \mathrm{i}/3)$ and recall from (2.1.1) the principal automorphism σ of $\mathfrak{sl}(3,\mathbf{C})$:

$$\sigma(H_i) = H_i, \qquad \sigma(E_i) = \eta E_i, \quad i = 0,1. \tag{3.2.15}$$

Equivalently, $\sigma = \exp(\pi \mathrm{i}\,\mathrm{ad}(H_1)/6)\mu$. Define the following elements of $\mathfrak{sl}(3,\mathbf{C})$:

$$\begin{aligned}&a_1 = \tfrac{1}{3}(E_0+E_1), \qquad a_2 = \tfrac{1}{2}(2F_0+F_1), \qquad a_3 = \tfrac{1}{3}(2E_0-E_1),\\ &a_4 = \tfrac{1}{4}(4F_0-F_1), \qquad a_5 = \tfrac{1}{2}[E_1,E_0], \qquad a_6 = \tfrac{1}{2}[F_0,F_1],\\ &a_7 = \frac{1}{2\sqrt{2}}H_1, \qquad a_8 = \frac{1}{2\sqrt{6}}\bigl[E_1,[E_1,E_0]\bigr],\end{aligned} \tag{3.2.16}$$

and let $\pi = (12)(34)(56) \in \mathcal{S}_8$. Then $\{a_i \mid i \in \{1,\dots,8\}\}$ and $\{a_{\pi(i)} \mid i \in \{1,\dots,8\}\}$ are σ-homogeneous dual bases of $\mathfrak{sl}(3,\mathbf{C})$ such that $\sigma(a_i) = \eta a_i$, $i = 1,3$, $\sigma(a_i) = \eta^5 a_i$, $i = 2,4$, $\sigma(a_5) = \eta^2 a_5$, $\sigma(a_6) = \eta^4 a_6$, $\sigma(a_7) = a_7$, $\sigma(a_8) = \eta^3 a_8$, and $\{a_1, a_2\}$, $\{a_2, a_1\}$ are σ-homogeneous dual bases of $\mathfrak{s}$. Consider now the following subalgebras of $\widetilde{\mathfrak{sl}(3,\mathbf{C})}$:

$$\hat{\mathfrak{s}} = \mathfrak{s} \otimes \mathbf{C}\big[t, t^{-1}\big] \oplus \mathbf{C}c, \qquad \tilde{\mathfrak{s}} = \hat{\mathfrak{s}} \rtimes \mathbf{C}d,$$
$$\tilde{\mathfrak{s}}' = [\tilde{\mathfrak{s}}, \tilde{\mathfrak{s}}], \qquad \tilde{\mathfrak{s}}'_{\pm} = \mathfrak{s} \otimes t^{\pm 1}\mathbf{C}\big[t^{\pm 1}\big]. \tag{3.2.17}$$

Notice that $\tilde{\mathfrak{s}}' = \tilde{\mathfrak{s}}'_+ \oplus \tilde{\mathfrak{s}}'_- \oplus \mathbf{C}c$ is a Heisenberg Lie subalgebra of $\hat{\mathfrak{s}}$, and form the induced irreducible $\hat{\mathfrak{s}}$-module (which is irreducible even as an $\tilde{\mathfrak{s}}'$-module)

$$M(2k+1) = U(\hat{\mathfrak{s}}) \otimes_{U(\mathfrak{s}\otimes\mathbf{C}[t]\oplus\mathbf{C}c)} \mathbf{C}, \tag{3.2.18}$$

where $\mathfrak{s} \otimes \mathbf{C}[t]$ acts trivially on $\mathbf{C}$ while c acts as multiplication by $2k+1$. Recall that $M(2k+1) \cong S(\tilde{\mathfrak{s}}'_-)$ as vector spaces, and that the action of $\hat{\mathfrak{s}}$ extends to $\tilde{\mathfrak{s}}$ by letting d act as the degree operator, so that $M(2k+1)$ acquires thereby a $\mathbf{Z}$-gradation.

Remark 3.2.3. Notice that $\mathfrak{s}_{(1)} = \mathbf{C}a_1$ and $\mathfrak{s}_{(5)} = \mathbf{C}a_2$, and let

$$\hat{\mathfrak{s}}[\sigma] = \mathfrak{s}_{(1)} \otimes t^{1/6}\mathbf{C}\big[t, t^{-1}\big] \oplus \mathfrak{s}_{(5)} \otimes t^{5/6}\mathbf{C}\big[t, t^{-1}\big] \oplus \mathbf{C}c. \tag{3.2.19}$$

Then $\hat{\mathfrak{s}}[\sigma] = \tilde{\mathfrak{s}}[\sigma]' = [\tilde{\mathfrak{s}}[\sigma], \tilde{\mathfrak{s}}[\sigma]]$, where $\tilde{\mathfrak{s}}[\sigma] \subset \widetilde{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$ is defined in (2.2.8).

Let now $V_{2k+1}(A_2^{(1)}) = L((2k+1)\Lambda_0; A_2^{(1)})$. When considered as a $\tilde{\mathfrak{s}}'$-module, $V_{2k+1}(A_2^{(1)})$ decomposes by [21, Theorem 1.7.3] as $V_{2k+1}(A_2^{(1)}) = M(2k+1) \otimes \Omega_{2k+1} \cong S(\tilde{\mathfrak{s}}'_-) \otimes \Omega_{2k+1}$, where Ω_{2k+1} is the vacuum subspace of $V_{2k+1}(A_2^{(1)})$ for the action of $\tilde{\mathfrak{s}}'$. Set

$$\omega' = \frac{1}{4(k+2)} \sum_{i=1}^{8} a_i(-1)a_{\pi(i)}(-1)\mathbf{1} \in V_{2k+1}\big(A_2^{(1)}\big), \tag{3.2.20}$$

$$\omega_1 = \frac{1}{2(2k+1)} \sum_{i=1}^{2} a_i(-1)a_{\pi(i)}(-1)\mathbf{1} \in M(2k+1). \tag{3.2.21}$$

By Theorems 3.1.4 and 3.1.5, $(V_{2k+1}(A_2^{(1)}), Y, \mathbf{1}, \omega')$ is a σ-rational VOA of rank $c(k) := 4(2k+1)/(k+2)$ and $\{(L_{k_0,2k+1}(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]), Y^\sigma) \mid k_0 \in \{0,1,\dots,k\}\}$ is the complete set of its simple σ-twisted modules, the map Y^σ being as in (3.1.5). On the other hand, $M(2k+1)$ is stable under $Y(a,z)|_{M(2k+1)}$ for all $a \in M(2k+1)$, and it is not difficult to prove that when equipped with the restricted map, $(M(2k+1), Y, \mathbf{1}, \omega_1)$ becomes a VOA of rank $\dim \mathfrak{s} = 2$ (cf., e.g., [9,24]). Moreover, $(M(\sigma; 2k+1), Y^\sigma)$ is a simple σ-twisted $M(2k+1)$-module, where $M(\sigma; 2k+1)$ is the irreducible $\tilde{\mathfrak{s}}[\sigma]'$-module defined in (2.2.9).

Let $\{\beta_1, \beta_2\} \subset \mathfrak{s}^*$ be the basis of the root system of $\mathfrak{sl}(3, \mathbf{C})$ with associated coroots $\{a_1 + a_2, \eta^2 a_1 - \eta a_2\}$. Form the root lattice $Q = \mathbf{Z}\beta_1 + \mathbf{Z}\beta_2$ and note that

$$\Omega_{2k+1} = \coprod_{\beta \in Q} \Omega_{2k+1}^{\beta}, \ \Omega_{2k+1}^{\beta} = \bigl\{v \in \Omega_{2k+1} \mid a(0)v = \beta(a)v \text{ for } a \in \mathfrak{s}\bigr\}. \tag{3.2.22}$$

Then

$$\begin{aligned} \Omega_{2k+1}^{0} &= \bigl\{v \in V_{2k+1}\bigl(A_2^{(1)}\bigr) \bigm| a(n)v = 0 \text{ for } a \in \mathfrak{s},\ n \geqslant 0\bigr\} \\ &= \bigl\{v \in V_{2k+1}\bigl(A_2^{(1)}\bigr) \bigm| a(n)v = 0 \text{ for } a \in M(2k+1),\ n \geqslant 0\bigr\}, \end{aligned} \tag{3.2.23}$$

and it follows from (3.2.20) and (3.2.21) that

$$\omega_2 := \omega' - \omega_1 \in \Omega_{2k+1}^{0}. \tag{3.2.24}$$

Let $Y(\omega', z) = \sum_{n\in\mathbf{Z}} L(n) z^{-n-2}$ and $Y(\omega_i, z) = \sum_{n\in\mathbf{Z}} L^i(n) z^{-n-2}$, $i = 1, 2$, so that $L(n) = L^1(n) + L^2(n)$. Then $L^1(-1)b = 0$ for every $b \in \Omega_{2k+1}^{0}$ by (3.2.23), hence $Y(L^2(-1)b, z) = Y(L(-1)b, z) = \frac{\mathrm{d}}{\mathrm{d}z} Y(b, z)$. Furthermore, (3.2.21) implies that $L(1)\omega_1 = 0$ and then by [31, Theorem 5.1] one gets that $[L^1(m), L^2(n)] = 0$ and

$$\begin{aligned} &\bigl[L^2(m), L^2(n)\bigr] \\ &\quad = (m-n)L^2(m+n) + \tfrac{1}{12}\bigl(m^3 - m\bigr)\delta_{m+n,0} c_2(k) \quad \text{for } m, n \in \mathbf{Z}, \end{aligned}$$

where $c_2(k) := c(k) - 2 = 2c_1(k)$. By (3.1.1) and (3.2.23) Ω_{2k+1}^{0} is closed under vertex operators so that when equipped with the restricted map, $(\Omega_{2k+1}^{0}, Y, \mathbf{1}, \omega_2)$ becomes a VOA of rank $c_2(k)$.

Remark 3.2.4. (i) Since $(\Omega_{2k+1}^{\beta}, Y)$, $\beta \in Q$, are inequivalent simple Ω_{2k+1}^{0}-modules, the VOA Ω_{2k+1}^{0} is not rational. It is also easy to see that Ω_{2k+1}^{0} is in fact a simple VOA.

(ii) It follows from [31, Theorem 5.2] that Ω_{2k+1}^{0} can equivalently be defined as the set $\{v \in V_{2k+1}(A_2^{(1)}) \mid L^1(-1)v = 0\}$. Using the terminology of [31], we may call Ω_{2k+1}^{0} the commutant of $M(2k+1)$ (the commutant construction corresponds to the coset construction in physics). Notice also that Ω_{2k+1}^{0} has no weight one vectors, as it can be seen directly from the structure of $N((2k+1)\Lambda_0)$ (cf. (A.3) in Appendix A for $k = 1$).

Fix $k_0 \in \{0, 1, \ldots, k\}$ and denote the vertex operators on $L_{k_0,2k+1}(\widehat{\mathfrak{sl}(3, \mathbf{C})}[\sigma])$ associated to ω' and ω_i, $i = 1, 2$, respectively by

$$Y^\sigma(\omega',z)=\sum_{n\in\mathbf{Z}}L(n)z^{-n-2},$$
$$Y^\sigma(\omega_i,z)=\sum_{n\in\mathbf{Z}}L^i(n)z^{-n-2},\quad i=1,2, \tag{3.2.25}$$

so that $L(n)=L^1(n)+L^2(n)$. Then $[L^1(m),L^2(n)]=0$, $m,n\in\mathbf{Z}$, and $\Omega_{k_0,2k+1}$ is stable under $Y^\sigma(a,z)$ for all $a\in\Omega^0_{2k+1}$ by (3.1.1). Let $v_{k_0}\in\Omega_{k_0,2k+1}$ be a highest weight vector of $L_{k_0,2k+1}(\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma])$. Recall that $L(0)$ induces a $\frac{1}{6}\mathbf{Z}$-gradation on this module and let $(\Omega_{k_0,2k+1})_s=\{v\in\Omega_{k_0,2k+1}\mid L^2(0)v=sv\}$, $s\in\mathbf{C}$. Since $L^1(0)$ acts as a scalar operator on $\Omega_{k_0,2k+1}$ (cf. (3.2.27) below), it follows that $(\Omega_{k_0,2k+1},Y^\sigma)$ is a $\frac{1}{6}\mathbf{Z}$-graded weak σ-twisted Ω^0_{2k+1}-module such that

$$\Omega_{k_0,2k+1}=\coprod_{n\in\frac{1}{6}\mathbf{N}}(\Omega_{k_0,2k+1})_{n+\lambda(k_0)},\quad \dim(\Omega_{k_0,2k+1})_{\lambda(k_0)}=1,$$
$$\dim(\Omega_{k_0,2k+1})_{n+\lambda(k_0)}<\infty,\quad n\in\tfrac{1}{6}\mathbf{N},$$

where $\lambda(k_0)\in\mathbf{Q}$ is the lowest weight of $\Omega_{k_0,2k+1}$ given by $L^2(0)v_{k_0}=\lambda(k_0)v_{k_0}$. Thus $(\Omega_{k_0,2k+1},Y^\sigma)$ is in fact a σ-twisted $(\Omega^0_{2k+1},Y,\mathbf{1},\omega_2)$-module whose q-trace is

$$\begin{aligned} f_{k_0,2k+1}(q) &:= \mathrm{tr}_{\Omega_{k_0,2k+1}}q^{L^2(0)-c_2(k)/24}\\ &= q^{\lambda(k_0)-c_2(k)/24}\sum_{n=0}^{\infty}\bigl(\dim(\Omega_{k_0,2k+1})_{n/6+\lambda(k_0)}\bigr)q^{n/6}\\ &= q^{\lambda(k_0)-c_2(k)/24}\dim_*\Omega_{k_0,2k+1}, \end{aligned}\tag{3.2.26}$$

where $\dim_*\Omega_{k_0,2k+1}$ is as in (2.2.11). In order to compute $\lambda(k_0)$, let us again denote the canonical generators of $\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$ as in Remark 2.1.1. Note that $H_1=\frac{1}{3}(h_1^\sigma-4h_0^\sigma)$, and that from Lemma 3.1.6 and the definitions of $\Omega_{k_0,2k+1}$ and ω_1 it follows that

$$L^1(0)|_{\Omega_{k_0,2k+1}}=\tfrac{5}{72}\,\mathrm{id}_{\Omega_{k_0,2k+1}}. \tag{3.2.27}$$

Applying Lemma 3.1.6 to each term in (3.2.20) one gets that

$$L(0)v_{k_0}=\frac{18k_0^2-12k_0k+2k^2+12k_0+41k+20}{144(k+2)}\,v_{k_0}. \tag{3.2.28}$$

Then (3.2.27) and (3.2.28) give

$$\lambda(k_0)=\frac{18k_0^2-12k_0k+2k^2+12k_0+31k}{144(k+2)},$$

so that by (3.2.11)

$$\lambda(k_0)-\frac{c_2(k)}{24}=\frac{1}{2}\biggl(h_{k_0,k}-\frac{\tilde c_1(k)}{24}\biggr)=\frac{1}{2}\biggl(h^\nu_{k_0,k}-\frac{c_1(k)}{24}\biggr). \tag{3.2.29}$$

Plugging (3.2.29) in (3.2.26) and comparing with (3.2.6) and (3.2.9), one concludes that

$$f_{k_0,2k+1}(q) = \chi_{k_0,k}\big(q^{1/2}\big) = \chi^{\nu}_{k_0,k}\big(q^{1/2}\big) \quad \text{for } k_0 \in \{0,1,\dots,k\}. \tag{3.2.30}$$

Summarizing, we proved the following theorem.

Theorem 3.2.5. *$(\Omega^0_{2k+1}, Y, \mathbf{1}, \omega_2)$ is a simple VOA of rank $c_2(k) = 2c_1(k)$. Moreover, for every $k_0 \in \{0,1,\dots,k\}$, $(\Omega_{k_0,2k+1}, Y^{\sigma})$ is a σ-twisted Ω^0_{2k+1}-module whose q-trace $f_{k_0,2k+1}(q)$ satisfies $f_{k_0,2k+1}(q) = \chi^{\nu}_{k_0,k}(q^{1/2})$, where $\chi^{\nu}_{k_0,k}(q)$ is as in Theorem* 3.2.1.

Remark 3.2.6. As in the $A_1^{(1)}$-case, one could at first consider $L_{k_0,2k+1}(A_2^{(2)})$ just as a simple μ-twisted $V_{2k+1}(A_2^{(1)})$-module, and then use Proposition 3.1.2 in order to modify ω' appropriately. Indeed, let $\widetilde{\omega}' = \omega' + \frac{1}{12}H_1(-2)\mathbf{1}$. It is easily checked that $(V_{2k+1}(A_2^{(1)}), Y, \mathbf{1}, \widetilde{\omega}')$ becomes a $\frac{1}{6}\mathbf{N}$-graded VOA of rank $\frac{2}{3}(2k+1)(4-k)/(k+2)$ such that its q-trace coincides up to a power of q with the rescaled $(4,1,1)$-specialization of $\mathrm{e}^{-(2k+1)\Lambda_0}\,\mathrm{ch}\,L((2k+1)\Lambda_0; A_2^{(1)})$. Then $\widetilde{\omega}'_1$ induces the rescaled principal gradation on $L_{k_0,2k+1}(A_2^{(2)})$ as needed. However, in this “picture” one does not get a representation of $\hat{\mathfrak{s}}[\sigma]$ on $L_{k_0,2k+1}(A_2^{(2)})$ by means of μ-twisted vertex operators parameterized by some elements of $V_{2k+1}(A_2^{(1)})$ (see also (A.2) in Appendix A). In order to achieve this, one would have to deform the μ-twisted vertex operators into σ-twisted vertex operators as in [25, Proposition 5.4], which reduces essentially to the viewpoint adopted above.

Let us now see how appropriate the structures of Theorems 3.2.1 and 3.2.5 are in the context of Problems 1 and 2. As shown in Appendix A for $k=1$, Ω^0_{2k+1} and $V_k(A_1^{(1)})$ may in fact be nonisomorphic even as $\mathbf{Z}$-graded vector spaces. However, it is important to notice that the q-traces $\chi^{\nu}_{k_0,k}(q)$ and $f_{k_0,2k+1}(q)$ are actually equal only up to the transformation $q \to q^{1/2}$. Moreover, both the ranks of the VOAs $V_k(A_1^{(1)})$ and Ω^0_{2k+1} and the orders of the automorphisms ν and σ differ by the same factor 2. Obviously, it would be preferable not to have these differences, if possible. This can actually be arranged by using the recently developed permutation orbifold theory [12], as we shall now explain.

Recall from (3.1.9) that the tensor product VOA $(V_k(A_1^{(1)})^{\otimes 2}, Y_{\otimes}, \mathbf{1}_{\otimes}, \omega_{\otimes})$ has rank $c_2(k) = 2c_1(k)$, and notice that the diagonal action of $\mathrm{Aut}(V_k(A_1^{(1)}))$ on $V_k(A_1^{(1)})^{\otimes 2}$ commutes with the action of the symmetric group $\mathcal{S}_2$ on $V_k(A_1^{(1)})^{\otimes 2}$. Let ψ be the generator of $\mathcal{S}_2$ and form the 6th order automorphism $\tau = \nu^2\psi \in \mathrm{Aut}(V_k(A_1^{(1)})^{\otimes 2})$. Given a ν-twisted $V_k(A_1^{(1)})$-module M, one may define a structure of τ-twisted $V_k(A_1^{(1)})^{\otimes 2}$-module on M in the following way. By [35, Proposition 2.1.1], the identity

$$\left(\exp\left(-\sum_{j\in\mathbf{Z}_+} C_j x^{j+1}\frac{\mathrm{d}}{\mathrm{d}x}\right)\right)\cdot x = x+\frac{x^2}{2} \tag{3.2.31}$$

uniquely determines a sequence $\{C_j\}_{j\in\mathbf{Z}_+}\in\mathbf{Q}^\infty$, and one defines the operator

$$\begin{aligned}\Delta_2(z) &= \exp\left(\sum_{j\in\mathbf{Z}_+} C_j z^{-j/2}L(j)\right)2^{-L(0)}z^{-\frac{1}{2}L(0)}\\ &\in \left(\operatorname{End} V_k\left(A_1^{(1)}\right)\right)\left[\!\left[z^{1/2},z^{-1/2}\right]\!\right],\end{aligned} \tag{3.2.32}$$

where $L(j)=\operatorname{Res}_z z^{j+1}Y(\omega,z)$, $j\in\mathbf{Z}_+$, and $2^{-L(0)}z^{-\frac{1}{2}L(0)}$ acts on each homogeneous subspace $V_k(A_1^{(1)})_n$ as $2^{-n}z^{-n/2}$, $n\in\mathbf{Z}$. Recall that $\eta=\exp(\pi\mathrm{i}/3)$ and let $r\in\{0,1,2\}$ and $a\in V_k(A_1^{(1)})$ be such that $\nu(a)=\eta^{-2r}a$. Define the following operators:

$$\begin{aligned}&Y^\tau(a\otimes\mathbf{1},z)=Y^\nu\left(\Delta_2(z)a,z^{1/2}\right)\in(\operatorname{End}M)\left[\!\left[z^{1/6},z^{-1/6}\right]\!\right],\\ &Y^\tau(\mathbf{1}\otimes a,z)=\eta^2\lim_{z^{1/6}\to\eta z^{1/6}}Y^\tau(a\otimes\mathbf{1},z)\in(\operatorname{End}M)\left[\!\left[z^{1/6},z^{-1/6}\right]\!\right].\end{aligned} \tag{3.2.33}$$

It was proved in [12] that $\{Y^\tau(a\otimes\mathbf{1},z),Y^\tau(\mathbf{1}\otimes b,z)\mid a,b\in V_k(A_1^{(1)})\}$ generates a local system of τ-twisted vertex operators in the sense of [25]. By using the theory of *loc. cit.*, it was further shown in [12, Sections 3 and 6] that the vertex map Y^τ extends to the whole space $V_k(A_1^{(1)})^{\otimes 2}$ in such a way that (M,Y^τ) becomes a τ-twisted $V_k(A_1^{(1)})^{\otimes 2}$-module. Then one can prove the following theorem.

Theorem 3.2.7. *The VOA $(V_k(A_1^{(1)})^{\otimes 2},Y_\otimes,\mathbf{1}_\otimes,\omega_\otimes)$ of rank $c_2(k)$ is τ-rational and its irreducible τ-twisted modules are exactly $(L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu]),Y^\tau)$, $k_0\in\{0,1,\ldots,k\}$. Moreover, the q-trace $\chi^\tau_{k_0,k}(q)$ of $(L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu]),Y^\tau)$ coincides with the q-trace $f_{k_0,2k+1}(q)$ of the σ-twisted Ω^0_{2k+1}-module $(\Omega_{k_0,2k+1},Y^\sigma)$, and one has*

$$\chi^\tau_{k_0,k}(q)=f_{k_0,2k+1}(q)=\chi^\nu_{k_0,k}\left(q^{1/2}\right),\quad k_0\in\{0,1,\ldots,k\}, \tag{3.2.34}$$

where $\chi^\nu_{k_0,k}(q)$ is as in Theorem 3.2.1.

Proof. The first statement follows from Theorems 6.8, 6.4, and 3.8 in [12], so that it remains to prove (3.2.34). Recall that $\omega_\otimes=\omega\otimes\mathbf{1}+\mathbf{1}\otimes\omega$ and write $Y^\tau(\omega_\otimes,z)=\sum_{n\in\mathbf{Z}}L^\tau(n)z^{-n-2}$. By (3.2.33) one has

$$Y^\tau(\omega_\otimes,z)=Y^\nu\left(\Delta_2(z)\omega,z^{1/2}\right)+\eta^2\lim_{z^{1/6}\to\eta z^{1/6}}Y^\nu\left(\Delta_2(z)\omega,z^{1/2}\right). \tag{3.2.35}$$

The constants C_j in (3.2.31) may be computed recurrently for small values of j. For instance, $C_1 = -1/2$ and $C_2 = 1/4$. Moreover, the Virasoro algebra commutation relations for the VOA $V_k(A_1^{(1)})$ imply that $L(2)\omega = \frac{1}{2}c_1(k)\mathbf{1}$ and $L(1)\omega = L(j)\omega = 0$ for $j \geqslant 3$ (cf. [23]). Then (3.2.32) yields

$$\Delta_2(z)\omega = \frac{z^{-1}}{4}\left(\omega + \frac{c_1(k)C_2}{2}z^{-1}\right) = \frac{z^{-1}}{4}\left(\omega + \frac{c_1(k)}{8}z^{-1}\right),$$

so that

$$Y^\nu\big(\Delta_2(z)\omega, z^{1/2}\big) = \frac{z^{-1}}{4}Y^\nu\big(\omega, z^{1/2}\big) + \frac{c_1(k)}{32}\,\mathrm{id}_{M_{k_0,k}}, \tag{3.2.36}$$

where $M_{k_0,k} := L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$. Plugging (3.2.36) in (3.2.35) and then extracting the coefficient of z^{-2} from (3.2.35) one gets that $L^\tau(0) = \frac{1}{2}L(0) + \frac{1}{16}c_1(k)\,\mathrm{id}_{M_{k_0,k}}$, and thus

$$L^\tau(0) - \frac{c_2(k)}{24}\,\mathrm{id}_{M_{k_0,k}} = \frac{1}{2}\left(L(0) - \frac{c_1(k)}{24}\,\mathrm{id}_{M_{k_0,k}}\right)$$

since $c_2(k) = 2c_1(k)$. It then follows from (3.2.9) that

$$\mathrm{tr}_{M_{k_0,k}}\, q^{L^\tau(0)-c_2(k)/24} = \mathrm{tr}_{M_{k_0,k}}\, q^{\frac{1}{2}(L(0)-c_1(k)/24)} = \chi^\nu_{k_0,k}\big(q^{1/2}\big),$$

which is the same as (3.2.34). □

Remark 3.2.8. (i) As used here, a ν-twisted $V_k(A_1^{(1)})$-module is the same as a ν^{-1}-twisted $V_k(A_1^{(1)})$-module in the sense of [12]. Note that $\nu^{-1} = \nu^2$ in our case.

(ii) An interesting similarity between the automorphisms τ and σ is that both of them are compositions of two commuting automorphisms of orders 3 and 2 in $\mathrm{Aut}(V_k(A_1^{(1)})^{\otimes 2})$ and $\mathrm{Aut}(\Omega^0_{2k+1})$, respectively (cf. (3.2.15)). Note also that in both cases the order 3 automorphism is inner and the involution is outer.

In view of Theorems 3.2.5 and 3.2.7, one may ask whether the VOAs $V_k(A_1^{(1)})^{\otimes 2}$ and Ω^0_{2k+1} are isomorphic. As shown in Appendix A for $k = 1$, this is not necessarily true. On the other hand, nothing was mentioned so far about the irreducibility of the σ-twisted Ω^0_{2k+1}-modules $\Omega_{k_0,2k+1}$, $k_0 \in \{0, 1, \ldots, k\}$. Note that in the level one case—which is excluded from the present discussion—the Ω^0_1-module $\Omega_{0,1}$ is trivially irreducible since $\dim \Omega_{0,1} = 1$, as it can be seen from (2.2.5) and (2.2.11). The general results on coset constructions and dual pairs for VOAs recently obtained in [14] seem to indicate that these σ-twisted Ω^0_{2k+1}-modules are indeed simple, but this is highly nontrivial for $k \geqslant 1$. We shall discuss such irreducibility questions in a different setting in Section 4.4.

Remark 3.2.9. (i) As a step in the direction suggested in Problem 2, one could investigate whether $L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$ and $\Omega_{k_0,2k+1}$ are isomorphic *Vir*-modules with central charge $c_2(k)$ (cf. Theorems 3.2.5 and 3.2.7). Note though that even if true, this would not provide a fully satisfactory answer to Problem 2 since $L_{k_0,k}(\widehat{\mathfrak{sl}(2,\mathbf{C})}[\nu])$ and $\Omega_{k_0,2k+1}$ are not simple *Vir*-modules.

(ii) Let V be a VOA with a finite order automorphism σ and denote by $A_\sigma(V)$ the twisted analogue of Zhu's algebra $A(V)$ (cf. [26,28]). Using the results of [28] one can actually show that $A_\tau(V_k(A_1^{(1)})^{\otimes 2}) \cong A_\nu(V_k(A_1^{(1)})) \cong A_\sigma(V_{2k+1}(A_2^{(1)}))$. However, this is more a reformulation of the fact that $|\mathcal{O}_1(k)| = |\mathcal{O}_2(k)| = k+1$ rather than a conceptual explanation to Theorem 2.2.1.

4. The GVOA Ω^A_{2k+1} and its action on the spaces in $\mathcal{O}_2(k)$

Continuing our study of Problem 1, we shall now embed Ω^0_{2k+1} into a larger structure, namely a simple GVOA which acts irreducibly on each of the spaces $\Omega_{k_0,2k+1}$ without altering the q-traces $f_{k_0,2k+1}(q)$ of (3.2.26) (Theorems 4.4.3 and 4.4.8). This new structure is defined as the quotient Ω^A_{2k+1} of the vacuum space Ω_{2k+1} by the action of a certain infinite abelian group A, and it acts on the spaces $\Omega_{k_0,2k+1}$ by means of quotient relative σ-twisted vertex operators (introduced in [15]) whose component operators generate an algebra that includes the σ-twisted $\mathcal{Z}$-algebra of [3]. This is done by using the diagonal action of $\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$ on the tensor product of $2k+1$ copies of the basic standard $\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$-module, together with the fact that each standard $\widehat{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$-module of level $2k+1$ can be isomorphically embedded into this tensor product (cf. Section 2.1). As we explain at the end of Section 4.4, the GVOA Ω^A_{2k+1} appears to be a more appropriate tool than the VOA Ω^0_{2k+1} for further investigating Problem 1. The main reason for that is the "equivalence theorem" [3, Theorem 5.5], which essentially establishes an equivalence between appropriately defined module categories for $\widetilde{\mathfrak{sl}(3,\mathbf{C})}[\sigma]$ and (an algebra that includes) the σ-twisted $\mathcal{Z}$-algebra.

The methods used below are based on [3,9,15,36], where a great many technical ingredients were used. Since the proofs of our main results in Section 4.4 require a careful analysis of most of these ingredients, we first recall in Sections 4.1–4.3 some of the constructions of *loc. cit.* and adapt them to our particular case. We refer to the above-mentioned papers for additional details.

Throughout this section we let $l = 2k+1$, where $k \in \mathbf{Z}_+$ is fixed, and we assume that $\mathfrak{g} = \mathfrak{sl}(3,\mathbf{C})$ with the bilinear form $\langle\cdot\,,\cdot\rangle$ normalized as in Section 2.1. The notations introduced below will partially overlap with some of the previous sections, as we need to work in the setting of [9,15].

4.1. The new setting

Recall from Section 3.2 the root basis $\{\beta_1, \beta_2\}$ and the root lattice Q of $\mathfrak{g}$, and let Φ be its root system. Let σ_1 be the reflection with respect to β_1 and μ be the automorphism of Q induced by the Dynkin diagram automorphism determined by $\mu\beta_1 = \beta_2$, $\mu\beta_2 = \beta_1$. Recall (3.2.13)–(3.2.16) and note that the linear extension of $\sigma_1\mu$ to $Q \otimes_{\mathbf{Z}} \mathbf{C}$ coincides with the restriction to $\mathfrak{s}$ of the principal automorphism σ of $\mathfrak{g}$, where $\mathfrak{s}$ and $Q \otimes_{\mathbf{Z}} \mathbf{C}$ are identified via the form $\langle\cdot\,,\cdot\rangle$. Since $\sigma_1\mu \equiv \sigma$ is a "twisted Coxeter element" of order 6, one has

$$\sum_{p=0}^{5} \sigma^p \alpha = 0, \quad \sum_{p=0}^{5} p\bigl\langle \sigma^p\alpha, \beta\bigr\rangle \equiv 0 \mod 6 \quad \text{for } \alpha, \beta \in Q. \tag{4.1.1}$$

Moreover, Q is a positive definite even lattice, $\langle\sigma^3\alpha, \alpha\rangle = -\langle\alpha, \alpha\rangle \in 2\mathbf{Z}$ for $\alpha \in Q$, and $\langle\cdot\,,\cdot\rangle$ is σ-invariant. Denote by $\langle\eta\rangle$ the cyclic group of order 6 generated by $\eta = \exp(\pi \mathrm{i}/3)$. Set

$$L = Q_1 \oplus \cdots \oplus Q_l, \tag{4.1.2}$$

where each Q_i is a copy of Q. We write $\alpha_i \in Q_i$ for the element corresponding to $\alpha \in Q$, and we extend the form $\langle\cdot\,,\cdot\rangle$ to L so that Q_i and Q_j are orthogonal if $i \neq j$.

Consider the σ-invariant alternating $\mathbf{Z}$-bilinear maps $c_0, c_0^\sigma : L \times L \to \mathbf{Z}/6\mathbf{Z}$ defined by $c_0(\alpha, \beta) = 3\langle\alpha, \beta\rangle + 6\mathbf{Z}$ and $c_0^\sigma(\alpha, \beta) = \sum_{p=0}^{5}(3 + p)\langle\sigma^p\alpha, \beta\rangle + 6\mathbf{Z}$, respectively, for $\alpha, \beta \in L$. Notice that $c_0^\sigma(\alpha, \beta) \equiv 0 \bmod 6$ for every $\alpha, \beta \in L$, and set

$$c(\alpha, \beta) = \eta^{c_0(\alpha,\beta)} = (-1)^{\langle\alpha,\beta\rangle}, \quad c_\sigma(\alpha, \beta) = \eta^{c_0^\sigma(\alpha,\beta)} = 1, \quad \alpha, \beta \in L. \tag{4.1.3}$$

Define also

$$\varepsilon_0(\alpha, \beta) = \langle\alpha + \sigma\alpha, \beta\rangle + 6\mathbf{Z}, \quad \varepsilon(\alpha, \beta) = \eta^{\varepsilon_0(\alpha,\beta)}, \quad \alpha, \beta \in L. \tag{4.1.4}$$

Then $\varepsilon_0(\cdot\,,\cdot)$ is a σ-invariant 2-cocycle on L satisfying

$$\varepsilon_0(\alpha, \beta) - \varepsilon_0(\beta, \alpha) = c_0(\alpha, \beta) - c_0^\sigma(\alpha, \beta) = c_0(\alpha, \beta), \tag{4.1.5}$$

and it follows from (4.1.1) together with $\sigma^3 = -1$ that $\varepsilon_0(\alpha, \beta) \equiv 0 \bmod 3$. Therefore, $\varepsilon(\cdot\,,\cdot)$ is σ-invariant and $\varepsilon(\alpha, \beta) \in \{\pm 1\}$ for $\alpha, \beta \in L$. Up to equivalence, the commutator maps c and c_σ uniquely determine two central extensions of L

$$1 \to \langle\eta\rangle \to \widehat{L} \xrightarrow{-} L \to 1 \quad \text{and} \quad 1 \to \langle\eta\rangle \to \widehat{L}_\sigma \xrightarrow{-} L \to 1 \tag{4.1.6}$$

by the commutator relations

$$\begin{aligned} &aba^{-1}b^{-1} = c\bigl(\bar{a}, \bar{b}\bigr), \quad a, b \in \widehat{L}, \quad \text{and} \\ &aba^{-1}b^{-1} = c_\sigma\bigl(\bar{a}, \bar{b}\bigr) = 1, \quad a, b \in \widehat{L}_\sigma, \end{aligned} \tag{4.1.7}$$

respectively. Notice that $\widehat{L}_\sigma$ is a split extension of L, so that $\widehat{L}_\sigma = \langle\eta\rangle \times L$. One has a set-theoretic identification between the groups $\widehat{L}$ and $\widehat{L}_\sigma$ such that the respective group multiplications $\times$ and $\times_\sigma$ satisfy

$$a \times b = \varepsilon\bigl(\bar{a}, \bar{b}\bigr) a \times_\sigma b. \tag{4.1.8}$$

Moreover, σ lifts to an automorphism $\hat{\sigma}$ of $\widehat{L}$ such that

$$\hat{\sigma}\eta = \eta, \qquad \overline{\hat{\sigma}a} = \sigma\bar{a} \quad \text{for } a \in \widehat{L}, \tag{4.1.9}$$

and such a lifting is unique up to multiplication by a lifting of the identity automorphism of L (cf. [21, Proposition 5.4.1]). Since $\varepsilon(\cdot\,,\cdot)$ is σ-invariant, one gets from (4.1.8) that $\hat{\sigma}$ is also an automorphism of $\widehat{L}_\sigma$ covering σ. Let

$$\begin{aligned} e: L &\to \widehat{L} \\ \alpha &\mapsto e_\alpha = (1, \alpha) \end{aligned} \tag{4.1.10}$$

be the section corresponding to the cocycle $\varepsilon_0(\cdot\,,\cdot)$; i.e., $\bar{e}_\alpha = \alpha$, $e_\alpha e_\beta = \varepsilon(\alpha, \beta)e_{\alpha+\beta}$, $\alpha, \beta \in L$. Note that the bilinearity of $\varepsilon_0(\cdot\,,\cdot)$ implies that $e_0 = 1$ $(= (1, 0))$.

Remark 4.1.1. (i) The lattice in (4.1.2) is denoted by L_0 in [9, Chapters 13 and 14], where one defines L to be the direct sum of l copies of the weight lattice of $\mathfrak{g}$ instead (so that L_0 is the dual lattice of L in $L \otimes_{\mathbf{Z}} \mathbf{C}$). For our purposes though, it will suffice to use only the root lattice Q.

(ii) The 2-cocycle $\varepsilon_0(\cdot\,,\cdot)$ is cohomologous to the one defined in [15, (2.13)], from which it differs by the 2-coboundary $L \times L \ni (\alpha, \beta) \mapsto 2\langle\alpha, \beta\rangle + 6\mathbf{Z} \in \mathbf{Z}/6\mathbf{Z}$.

Applying the above discussion to a single copy of the root lattice Q, one gets from [21, Chapter 6] and [3, Section 8] that there exist $x_\alpha \in \mathfrak{g}$ $(\alpha \in \Phi)$ such that $\mathfrak{g} = \mathfrak{s} \oplus \coprod_{\alpha\in\Phi} \mathbf{C}x_\alpha$ and

$$\sigma x_\alpha = x_{\sigma\alpha}, \qquad \mathfrak{s}' = 0, \qquad [h, x_\alpha] = \langle h, \alpha\rangle x_\alpha,$$
$$[x_\alpha, x_\beta] = \begin{cases} \varepsilon(\alpha, -\alpha)\alpha & \text{if } \alpha + \beta = 0, \\ \varepsilon(\alpha, \beta)x_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ 0 & \text{if } \alpha + \beta \notin \Phi \cup \{0\}, \end{cases}$$

for $h \in \mathfrak{s}$ and $\alpha, \beta \in \Phi$. It is well known that the bilinear form $\langle\cdot\,,\cdot\rangle$ of $\mathfrak{g}$ satisfies

$$\langle h, x_\alpha\rangle = 0, \qquad \langle x_\alpha, x_\beta\rangle = \begin{cases} \varepsilon(\alpha, -\alpha) & \text{if } \alpha + \beta = 0, \\ 0 & \text{if } \alpha + \beta \neq 0, \end{cases}$$

for $\alpha, \beta \in \Phi$ and $h \in \mathfrak{s}$. Recall the σ-decomposition $\mathfrak{g} = \coprod_{j\in\mathbf{Z}/6\mathbf{Z}} \mathfrak{g}_{(j)}$ and let $\mathfrak{g}_{(n)} = \mathfrak{g}_{(n \bmod 6)}$ for $n \in \mathbf{Z}$. Note also that $\mathfrak{s} = \mathfrak{s}_{(1)} \oplus \mathfrak{s}_{(5)}$ (cf. (3.2.19)). Let further $x_{(n)}$ denote the projection of $x \in \mathfrak{g}$ on $\mathfrak{g}_{(n)}$, $n \in \mathbf{Z}$, and set

$$x(z) = \sum_{n\in\mathbf{Z}} \bigl(x \otimes t^n\bigr)z^{-n-1}, \qquad x(\sigma; z) = \sum_{n\in\frac{1}{6}\mathbf{Z}} \bigl(x_{(6n)} \otimes t^n\bigr)z^{-n-1}. \tag{4.1.11}$$

We denote by $\langle\cdot\,,\cdot\rangle$ as well the bilinear form induced on the space

$$\mathfrak{h} = L \otimes_{\mathbf{Z}} \mathbf{C}. \tag{4.1.12}$$

Let now $\mathfrak{h}_*$ be a subspace of $\mathfrak{h}$ on which $\langle\cdot\,,\cdot\rangle$ remains nonsingular, so that

$$\mathfrak{h} = \mathfrak{h}_* \oplus \mathfrak{h}_*^{\perp}, \tag{4.1.13}$$

where $^{\perp}$ denotes the orthogonal complement. We also assume that

$$\sigma\mathfrak{h}_* = \mathfrak{h}_* \tag{4.1.14}$$

and write

$$\begin{array}{l} \mathfrak{h} \to \mathfrak{h}_*^{\perp} \\ h \mapsto h' \end{array} \quad \text{and} \quad \begin{array}{l} \mathfrak{h} \to \mathfrak{h}_*, \\ h \mapsto h'' \end{array} \tag{4.1.15}$$

for the projection maps to $\mathfrak{h}_*^{\perp}$ and $\mathfrak{h}_*$, respectively. By (4.1.14), these maps commute with the action of σ. In Sections 4.2–4.3 we shall work with an arbitrary space $\mathfrak{h}_*$ satisfying (4.1.13)–(4.1.14). This flexibility for the choice of $\mathfrak{h}_*$ will allow us to recover the usual (unrelativized) untwisted and twisted vertex operators by taking $\mathfrak{h}_* = 0$, in which case the index $_*$ will be removed from the notation. In Section 4.4 though we shall specify $\mathfrak{h}_*$ to be the image of $\mathfrak{s}$ under its diagonal embedding in $\mathfrak{h}$.

4.2. *Relative untwisted vertex operators*

Form the induced $\widehat{L}$-module and $\mathbf{C}$-algebra $\mathbf{C}\{L\} = \mathbf{C}[\widehat{L}] \otimes_{\mathbf{C}[\langle\eta\rangle]} \mathbf{C} \cong \mathbf{C}[L]$ (linearly), where $\mathbf{C}[\cdot]$ denotes the group algebra and η acts on $\mathbf{C}$ as multiplication by η (here $\langle\eta\rangle$ is understood as an abstract group disjoint from $\mathbf{C}^{\times}$). For $a \in \widehat{L}$, let $\iota(a) = a \otimes 1$ be the image of a in $\mathbf{C}\{L\}$. Then the action of $\widehat{L}$ on $\mathbf{C}\{L\}$ and the product in $\mathbf{C}\{L\}$ are given by

$$a \cdot \iota(b) = \iota(a)\iota(b) = \iota(ab), \qquad \eta \cdot \iota(b) = \eta\iota(b) \tag{4.2.1}$$

for $a, b \in \widehat{L}$. We endow $\mathbf{C}\{L\}$ with the $\mathbf{C}$-gradation determined by

$$\operatorname{wt}\big(\iota(a)\big) = \tfrac{1}{2}\big\langle \bar{a}', \bar{a}'\big\rangle \quad \text{for } a \in \widehat{L}, \tag{4.2.2}$$

and we define a grading-preserving action of $\mathfrak{h}$ on $\mathbf{C}\{L\}$ by

$$h \cdot \iota(a) = \big\langle h', \bar{a}\big\rangle \iota(a) \tag{4.2.3}$$

for $h \in \mathfrak{h}$. The automorphism $\hat{\sigma}$ of $\widehat{L}$ acts canonically and in a grading-preserving fashion on $\mathbf{C}\{L\}$ such that $\hat{\sigma}\iota(a) = \iota(\hat{\sigma}a)$, $a \in \widehat{L}$, and $\hat{\sigma}(\iota(a)\iota(b)) = \hat{\sigma}(a \cdot \iota(b)) = \hat{\sigma}(a) \cdot \hat{\sigma}\iota(b) = \hat{\sigma}\iota(a)\hat{\sigma}\iota(b)$. Then $\mathfrak{h}$ acts as algebra derivations and $\hat{\sigma}(h \cdot \iota(a)) = \sigma(h) \cdot \hat{\sigma}\iota(a)$, $h \in \mathfrak{h}$, $a \in \widehat{L}$. Define also an action $z^h \cdot \iota(a) = z^{\langle h', \bar{a}\rangle}\iota(a)$ for $h \in \mathfrak{h}$, $a \in \widehat{L}$, so that $\hat{\sigma}(z^h \cdot \iota(a)) = z^{\sigma(h)} \cdot \hat{\sigma}\iota(a)$.

Form the algebras

$$\begin{aligned} &\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbf{C}\big[t, t^{-1}\big] \oplus \mathbf{C}c, \qquad \tilde{\mathfrak{h}} = \hat{\mathfrak{h}} \rtimes \mathbf{C}d, \\ &\tilde{\mathfrak{h}}' = \big[\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}\big], \qquad \tilde{\mathfrak{h}}'_{\pm} = \mathfrak{h} \otimes t^{\pm 1}\mathbf{C}\big[t^{\pm 1}\big], \end{aligned} \tag{4.2.4}$$

and notice that $\tilde{\mathfrak{h}}' = \tilde{\mathfrak{h}}'_+ \oplus \tilde{\mathfrak{h}}'_- \oplus \mathbf{C}c$ is a Heisenberg Lie subalgebra of the affine Lie algebra $\tilde{\mathfrak{h}}$. On $\hat{\mathfrak{h}}$ we define a weight gradation associated with $\mathfrak{h}_*$ by

$$\mathrm{wt}\big(x \otimes t^m\big) = 0, \qquad \mathrm{wt}\big(y \otimes t^n\big) = -n, \qquad \mathrm{wt}(c) = 0 \tag{4.2.5}$$

for $x \in \mathfrak{h}_*$, $y \in \mathfrak{h}_*^{\perp}$, and $m, n \in \mathbf{Z}$. By analogy with (3.2.18), we consider the induced irreducible $\tilde{\mathfrak{h}}'$-and $\hat{\mathfrak{h}}$-module $M(1,\mathfrak{h}) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbf{C}[t]\oplus\mathbf{C}c)} \mathbf{C} \cong S(\tilde{\mathfrak{h}}'_-)$ (linearly), where $\mathfrak{h}\otimes\mathbf{C}[t]$ acts trivially on $\mathbf{C}$ and c acts as 1. It follows from (4.2.5) that $M(1,\mathfrak{h})$ is $\mathbf{Z}$-graded so that $\mathrm{wt}(1) = 0$. Moreover, σ acts in a natural grading-preserving way on $\hat{\mathfrak{h}}$ (fixing c) and on $M(1,\mathfrak{h})$. Set

$$V_L = M(1,\mathfrak{h}) \otimes_{\mathbf{C}} \mathbf{C}\{L\} \cong S\big(\tilde{\mathfrak{h}}'_-\big) \otimes_{\mathbf{C}} \mathbf{C}[L] \quad \text{(linearly)}, \tag{4.2.6}$$

and endow V_L with the tensor product $\mathbf{C}$-gradation. In particular, $\mathrm{wt}(\iota(1)) = 0$, $\mathbf{C}\{L\}$ being naturally identified with $1 \otimes \mathbf{C}\{L\}$. Then $\widehat{L}$, $\tilde{\mathfrak{h}}'$, $\mathfrak{h}$, z^h $(h \in \mathfrak{h})$ act naturally on V_L by acting either on $M(1,\mathfrak{h})$ or $\mathbf{C}\{L\}$ as indicated above. The automorphism $\hat{\sigma}$ acts in a grading-preserving way on V_L via $\sigma \otimes \hat{\sigma}$, and this action is compatible with the other actions (cf. [15, (3.23)–(3.25)]). The relations

$$\begin{aligned} &i(h) = \big(h_1(-1) + \cdots + h_l(-1)\big) \cdot \iota(1), \quad h \in \mathfrak{s}, \quad \text{and} \\ &i(x_\alpha) = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_l}), \quad \alpha \in \Phi, \end{aligned} \tag{4.2.7}$$

define a linear injection i from $\mathfrak{g}$ to V_L. For $\alpha \in \mathfrak{h}$, $n \in \mathbf{Z}$, we denote by $\alpha(n)$ the operator on V_L determined by $\alpha \otimes t^n$, and we let $\alpha(z) = \sum_{n\in\mathbf{Z}} \alpha(n) z^{-n-1}$.

We can now introduce the relative untwisted vertex operators of [9]. These operators are parameterized by elements of the space V_L on which they also act, and they are defined relative to $\mathfrak{h}_*$. We use a "normal ordering" procedure $\mathop{\circ}\limits^{\circ} \cdot \mathop{\circ}\limits^{\circ}$ to signify that the enclosed expression is to be reordered if necessary so that all the operators $\alpha(n)$ $(\alpha \in \mathfrak{h}, n < 0)$ and $a \in \widehat{L}$ are to be placed to the left of all the operators $\alpha(n)$ and z^α $(\alpha \in \mathfrak{h}, n \geqslant 0)$ before the expression is evaluated. Using an obvious formal integration notation, we set

$$Y_*^{(l)}\big(\iota(a), z\big) = Y_*^{(l)}(a, z) = \mathop{\circ}\limits^{\circ} \exp\left[\int \big(\bar{a}'(z) - \bar{a}'(0) z^{-1}\big)\right] a z^{\bar{a}'} \mathop{\circ}\limits^{\circ}, \quad a \in \widehat{L}. \tag{4.2.8}$$

If $v \in V_L$ is of the form $v = \alpha_1(-n_1) \cdots \alpha_j(-n_j) \cdot \iota(a)$, $a \in \widehat{L}$, $\alpha_k \in \mathfrak{h}$, $n_k \in \mathbf{Z}_+$, $1 \leqslant k \leqslant j$, we define

$$\begin{aligned} Y_*^{(l)}(v, z) = {} & \mathop{\circ}\limits^{\circ} \left[\frac{1}{(n_1 - 1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_1 - 1} \alpha_1'(z)\right] \cdots \left[\frac{1}{(n_j - 1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_j - 1} \alpha_j'(z)\right] \\ & \times Y_*^{(l)}(a, z) \mathop{\circ}\limits^{\circ}, \end{aligned} \tag{4.2.9}$$

and we extend this definition to V_L by linearity. One gets a well-defined linear map

$$\begin{aligned} V_L &\to (\operatorname{End} V_L)\{z\} \\ v &\mapsto Y_*^{(l)}(v,z) = \sum_{n\in\mathbf{C}} v_n z^{-n-1}, \quad v_n \in \operatorname{End} V_L, \end{aligned} \tag{4.2.10}$$

where for any vector space W, $W\{z\}$ denotes the linear space of W-valued formal series in z. The case $\mathfrak{h}_* = 0$ leads to the (ordinary) untwisted vertex operators as defined in [21, Chapter 8]. In this case, the operators $Y_*^{(l)}(v,z)$ are denoted just by $Y^{(l)}(v,z)$. We refer to [9,15] for a detailed discussion of the properties satisfied by the operators $Y_*^{(l)}(v,z)$. One has in particular

$$\big[\tilde{\mathfrak{h}}'_*, Y_*^{(l)}(v,z)\big] = 0 \quad \text{for } v \in V_L, \tag{4.2.11}$$

where $\tilde{\mathfrak{h}}'_* = [\tilde{\mathfrak{h}}_*, \tilde{\mathfrak{h}}_*]$ is the Heisenberg algebra associated with the abelian Lie algebra $\mathfrak{h}_*$ (see (4.2.4)). Notice that when applied to a single copy of Q in the case $\mathfrak{h}_* = 0$, the above construction yields a well-defined linear map

$$\begin{aligned} V_Q &\to (\operatorname{End} V_Q)\big[\!\big[z, z^{-1}\big]\!\big] \\ v &\mapsto Y^{(1)}(v,z) = \sum_{n\in\mathbf{Z}} v_n z^{-n-1}, \end{aligned}$$

the lattice Q being even. Furthermore, using the notation

$$\begin{aligned} &\mathfrak{s}_i = Q_i \otimes_{\mathbf{Z}} \mathbf{C}, \qquad \hat{\mathfrak{s}}_i = \mathfrak{s}_i \otimes \mathbf{C}\big[t, t^{-1}\big] \oplus \mathbf{C}c, \\ &\tilde{\mathfrak{s}}'_{i\pm} = \mathfrak{s}_i \otimes t^{\pm 1}\mathbf{C}\big[\!\big[t^{\pm 1}\big]\!\big], \quad 1 \leqslant i \leqslant l, \end{aligned} \tag{4.2.12}$$

one gets from (4.1.2) that $\mathbf{C}\{L\} \cong \bigotimes_{i=1}^{l} \mathbf{C}\{Q_i\}$ and $S(\tilde{\mathfrak{h}}'_-) \cong \bigotimes_{i=1}^{l} S(\tilde{\mathfrak{s}}'_{i_-})$ as vector spaces, and thus $V_L \cong \bigotimes_{i=1}^{l} V_{Q_i}$ linearly (cf. (4.2.6)). For $1 \leqslant j \leqslant l$ define a linear injection i_j by

$$\begin{aligned} i_j : \mathfrak{g} &\to V_L \\ h &\mapsto h_j(-1)\cdot \iota(1), \quad h \in \mathfrak{s}, \\ x_\alpha &\mapsto \iota(e_{\alpha_j}), \quad \alpha \in \Phi, \end{aligned} \tag{4.2.13}$$

so that $i(x) = \sum_{j=1}^{l} i_j(x)$ for $x \in \mathfrak{g}$ by (4.2.7). Then from [9, Chapter 13] and [21, Section 7.2] one gets:

Theorem 4.2.1. *The linear map $\rho : \hat{\mathfrak{g}} \to \operatorname{End} V_Q$ given by*

$$\rho(c) = 1, \qquad \rho\big(x(z)\big) = Y^{(1)}\big(i(x), z\big), \quad x \in \mathfrak{g}, \tag{4.2.14}$$

defines a level 1 $\hat{\mathfrak{g}}$*-module structure on V_Q such that V_Q is thereby isomorphic to the basic module $L(\Lambda_0; \hat{\mathfrak{g}})$. Moreover, the linear map $\pi : \hat{\mathfrak{g}} \to \operatorname{End} V_L$ determined by*

$$\pi(c) = l, \qquad \pi\big(x(z)\big) = Y^{(l)}\big(i(x), z\big), \quad x \in \mathfrak{g}, \tag{4.2.15}$$

defines a level l $\hat{\mathfrak{g}}$-module structure on V_L such that the representation (4.2.15) *is isomorphic to the factor product of the representations*

$$\pi_j : \hat{\mathfrak{g}} \to \operatorname{End} V_{Q_j}, \quad 1 \leqslant j \leqslant l, \tag{4.2.16}$$

given by $\pi_j(c) = 1$, $\pi_j(x(z)) = Y^{(1)}(i_j(x), z)$, $x \in \mathfrak{g}$.

Remark 4.2.2. (i) The representations (4.2.14)–(4.2.16) extend to $\tilde{\mathfrak{g}}$ by letting d act as the degree operator (cf. (4.2.2) and (4.2.5) for $\mathfrak{h}_* = 0$), in which case (4.2.14) is the well-known Frenkel–Kac–Segal construction of the affine Lie algebra $\tilde{\mathfrak{g}}$.

(ii) Under the isomorphism in the second part of Theorem 4.2.1, the action of $Y^{(l)}(i_j(x), z)$ on V_L coincides with the action of $1 \otimes \cdots \otimes 1 \otimes Y^{(1)}(i_j(x), z) \otimes 1 \otimes \cdots \otimes 1$, where the nontrivial factor and is in the jth position.

By [9, Chapter 13], V_L is a completely reducible $\hat{\mathfrak{g}}$-module and its $\hat{\mathfrak{g}}$-submodule $U(\hat{\mathfrak{g}}) \cdot \iota(1)$ is isomorphic to the standard $\hat{\mathfrak{g}}$-module $L(l\Lambda_0; \hat{\mathfrak{g}})$, so that we may view $L(l\Lambda_0; \hat{\mathfrak{g}})$ as a subspace of V_L.

Corollary 4.2.3. *On the subspace $L(l\Lambda_0; \hat{\mathfrak{g}})$ of V_L the map $Y^{(l)}(\cdot, z)$ of* (4.2.15) *coincides with the map $Y(\cdot, z)$ of Theorems* 3.1.4 *and* 3.2.5.

Proof. Let $x \in \mathfrak{g}$, $n \in \mathbf{Z}$, and denote by $x(n)$ the operator $\pi(x \otimes t^n)$ acting on V_L. Then the map i can be rewritten as $i(x) = x(-1) \cdot \iota(1)$, and by comparing the expression for $x(z)$ in (4.1.11) with the generic expansion $Y^{(l)}(i(x), z) = \sum_{n\in\mathbf{Z}} x_n z^{-n-1}$ (cf. (4.2.10)) we see that in fact $x_n = x(n)$. For $x \in \mathfrak{g}$ one has therefore that $Y^{(l)}(i(x), z) = Y(x(-1)\mathbf{1}, z)$, where $\mathbf{1} = 1 \otimes \iota(1)$ and Y is the map defined in (3.1.4). Furthermore, from $L(l\Lambda_0; \hat{\mathfrak{g}}) \cong U(\hat{\mathfrak{g}}) \cdot \iota(1)$ one gets that $L(l\Lambda_0; \hat{\mathfrak{g}})$ is generated by the set $\{i(x) \mid x \in \mathfrak{g}\} \cup \{\mathbf{1}\}$ through the map $Y^{(l)}(\cdot, z)$, which by construction satisfies the same iterate formula as the map $Y(\cdot, z)$ (cf. (3.1.2)). This proves the corollary. □

4.3. Relative twisted vertex operators

We now concentrate on the twisted counterpart of the above constructions. Let again $\mathfrak{h}_* \subset \mathfrak{h}$ be as in (4.1.13)–(4.1.15) and recall from (4.1.6)–(4.1.8) the central extension $\widehat{L}_\sigma$ of L and the set-theoretic identification between the groups $\widehat{L}$ and $\widehat{L}_\sigma$. Notice that $1 - \sigma = \sigma^{-1}$. It follows from [36, Proposition 6.1] (see also [1]) that there exists a unique homomorphism $\psi : \widehat{L}_\sigma \to \mathbf{C}^\times$ such that $\psi(\eta) = \eta$ and L maps to the identity. Let $T = \mathbf{C}_\psi$ be the one-dimensional $\widehat{L}_\sigma$-module affording ψ, and give T the trivial $\mathbf{C}$-gradation. We let $\hat{\sigma}$ act on T as a grading-preserving linear automorphism (cf. (4.1.8)–(4.1.9)). By Remark 3.2.3, the σ-decomposition of $\mathfrak{h}$ is given by $\mathfrak{h} = \coprod_{n\in\mathbf{Z}/6\mathbf{Z}} \mathfrak{h}_{(n)} = \mathfrak{h}_{(1)} \oplus \mathfrak{h}_{(5)}$. Let $\mathfrak{h}_{(n)} =$

$\mathfrak{h}_{(n \bmod 6)}$, $n \in \mathbf{Z}$, and form the σ-twisted affine Lie algebra $\tilde{\mathfrak{h}}[\sigma] = \hat{\mathfrak{h}}[\sigma] \rtimes \mathbf{C}d$ and its subalgebras

$$\hat{\mathfrak{h}}[\sigma] = \bigoplus_{n\in\frac{1}{6}\mathbf{Z}} \mathfrak{h}_{(6n)} \otimes t^n \oplus \mathbf{C}c, \qquad \tilde{\mathfrak{h}}[\sigma]'_{\pm} = \bigoplus_{n\in\frac{1}{6}\mathbf{Z},\, \pm n>0} \mathfrak{h}_{(6n)} \otimes t^n, \quad (4.3.1)$$

so that $\hat{\mathfrak{h}}[\sigma] = \tilde{\mathfrak{h}}[\sigma]'_+ \oplus \tilde{\mathfrak{h}}[\sigma]'_- \oplus \mathbf{C}c$. Then $\tilde{\mathfrak{h}}[\sigma]' := [\tilde{\mathfrak{h}}[\sigma], \tilde{\mathfrak{h}}[\sigma]]$ is a Heisenberg Lie algebra and $\tilde{\mathfrak{h}}[\sigma]' = \hat{\mathfrak{h}}[\sigma]$. On $\hat{\mathfrak{h}}[\sigma]$ we define the weight gradation associated with $\mathfrak{h}_*$ by $\mathrm{wt}(x \otimes t^m) = 0$, $\mathrm{wt}(y \otimes t^n) = -n$, $\mathrm{wt}(c) = 0$, $m, n \in \frac{1}{6}\mathbf{Z}$, $x \in \mathfrak{h}_{*(6m)}$, $y \in \mathfrak{h}^{\perp}_{*(6n)}$ (cf. (4.2.5)). Form the induced irreducible $\hat{\mathfrak{h}}[\sigma]$-module $M(\sigma; 1, \mathfrak{h}) = U(\hat{\mathfrak{h}}[\sigma]) \otimes_{U(\bigoplus_{n\geqslant 0} \mathfrak{h}_{(6n)} \oplus \mathbf{C}c)} \mathbf{C} \cong S(\tilde{\mathfrak{h}}[\sigma]'_-)$ (linearly), where $\bigoplus_{n\geqslant 0} \mathfrak{h}_{(6n)}$ acts trivially on $\mathbf{C}$ and c acts as 1. Then $M(\sigma; 1, \mathfrak{h})$ has a natural $\frac{1}{6}\mathbf{Z}$-grading which is compatible with the action of $\hat{\mathfrak{h}}[\sigma]$ and such that $\mathrm{wt}(1) = 0$. Moreover, σ acts in a grading-preserving way on $\hat{\mathfrak{h}}[\sigma]$ (fixing c) and on $M(\sigma; 1, \mathfrak{h})$ (as an algebra isomorphism). Let $\hat{\mathfrak{h}}_*[\sigma]_-$ and $\hat{\mathfrak{h}}^{\perp}_*[\sigma]_-$ be defined as in (4.3.1) and set

$$\begin{aligned} V_L^T &= M(\sigma; 1, \mathfrak{h}) \otimes_{\mathbf{C}} T \cong S\big(\tilde{\mathfrak{h}}[\sigma]'_-\big) \\ &\cong S\big(\hat{\mathfrak{h}}_*[\sigma]_-\big) \otimes S\big(\hat{\mathfrak{h}}^{\perp}_*[\sigma]_-\big) \quad \text{(linearly)}, \end{aligned} \qquad (4.3.2)$$

(cf. (4.1.13)). Using the gradations of $M(\sigma; 1, \mathfrak{h})$ and T, we see that V_L^T is naturally $\mathbf{Q}$-graded. Clearly, $\hat{\mathfrak{h}}[\sigma]$ and $\widehat{L}_\sigma$ act on V_L^T by acting either on $M(\sigma; 1, \mathfrak{h})$ or T, and $\hat{\sigma}$ extends to a linear automorphism of V_L^T such that $\hat{\sigma}(u \otimes t) = \sigma(u) \otimes \hat{\sigma}(t) = t\sigma(u) \otimes 1$ for $u \in M(\sigma; 1, \mathfrak{h})$, $t \in T$.

For $\alpha \in \mathfrak{h}$ let $\alpha_{(6n)}(n)$ be the operator on $M(\sigma; 1, \mathfrak{h})$ corresponding to $\alpha_{(6n)} \otimes t^n$, $n \in \frac{1}{6}\mathbf{Z}$. Set

$$\alpha(\sigma; z) = \sum_{n\in\frac{1}{6}\mathbf{Z}} \alpha_{(6n)}(n) z^{-n-1}. \qquad (4.3.3)$$

We shall also use the scalar function

$$\tau(\alpha) = 2^{-\langle\alpha,\alpha\rangle}\big(1-\eta^{-1}\big)^{\langle\sigma\alpha,\alpha\rangle}\big(1-\eta^{-2}\big)^{\langle\sigma^2\alpha,\alpha\rangle} \quad \text{for } \alpha \in L, \qquad (4.3.4)$$

which obviously satisfies $\tau(\sigma\alpha) = \tau(\alpha)$. Given $a \in \widehat{L}$, one defines the relative σ-twisted vertex operator $Y_*^{\sigma,(l)}(\iota(a), z)$ acting on V_L^T as follows: for $\alpha \in \mathfrak{h}$ introduce first the operators

$$E^{\pm}_{*(1)}(\alpha, z) = \exp\left[\sum_{n\in\frac{1}{6}\mathbf{Z},\, \pm n>0} \frac{\alpha'_{(6n)}(n)}{n} z^{-n}\right] \in \big(\mathrm{End}\, S\big(\hat{\mathfrak{h}}^{\perp}_*[\sigma]_-\big)\big)\big[\!\big[z^{\pm 1/6}\big]\!\big] \qquad (4.3.5)$$

(which by (4.3.2) is a subspace of $(\mathrm{End}\, V_L^T)\{z\}$), and then set

$$\begin{aligned} Y_*^{\sigma,(l)}\big(\iota(a), z\big) &= Y_*^{\sigma,(l)}(a, z) \\ &= 6^{-\langle \bar{a}', \bar{a}' \rangle/2} \tau\big(\bar{a}'\big) {}^\circ_\circ \exp\left(\int \bar{a}'(\sigma; z)\right) a z^{-\langle \bar{a}', \bar{a}' \rangle/2} {}^\circ_\circ \\ &= 6^{-\langle \bar{a}', \bar{a}' \rangle/2} \tau\big(\bar{a}'\big) E^{-}_{*(1)}(-\bar{a}, z) E^{+}_{*(1)}(-\bar{a}, z) a z^{-\langle \bar{a}', \bar{a}' \rangle/2}, \end{aligned} \tag{4.3.6}$$

where a is viewed as an element of $\widehat{L}_\sigma$ by means of the set-theoretic identification between $\widehat{L}$ and $\widehat{L}_\sigma$. For $\alpha_i \in \mathfrak{h}$, $n_i \in \mathbf{Z}_+$, $1 \leqslant i \leqslant j$, and $v = \alpha_1(-n_1) \cdots \alpha_j(-n_j) \cdot \iota(a) \in V_L$, set

$$\begin{aligned} W_*(v, z) &= {}^\circ_\circ \left[\frac{1}{(n_1 - 1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_1 - 1} \alpha_1'(\sigma; z)\right] \\ &\quad \cdots \left[\frac{1}{(n_j - 1)!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_j - 1} \alpha_j'(\sigma; z)\right] Y_*^{\sigma,(l)}(a, z) {}^\circ_\circ \end{aligned} \tag{4.3.7}$$

(the right-hand side being an operator on V_L^T), and then extend this definition to all $v \in V_L$ by linearity. Let $\{\gamma_1, \ldots, \gamma_d\}$ be an orthonormal basis of $\mathfrak{h}_*^\perp$ and define the constants $c_{mni} \in \mathbf{C}$, $m, n \in \mathbf{N}$, $i \in \{0, 1, \ldots, 5\}$, as in [15, (4.41)]. In particular, $c_{00i} = 0$ for all i. The following operator is independent of the choice of orthonormal basis in $\mathfrak{h}_*^\perp$ (cf. [15, (4.42)], [21, Section 9.2]):

$$\Delta_{z*} = \sum_{m,n \geqslant 0} \sum_{i=0}^{5} \sum_{j=1}^{d} c_{mni} \big(\sigma^{-i} \gamma_j\big)(m) \gamma_j(n) z^{-m-n} \in (\operatorname{End} V_L) [\![z^{-1}]\!]. \tag{4.3.8}$$

Since $c_{00i} = 0$ for all i, $\exp(\Delta_{z*})$ is well defined on V_L and $\exp(\Delta_{z*})v \in V_L[z^{-1}]$ for $v \in V_L$. The relative σ-twisted vertex operator $Y_*^{\sigma,(l)}(v, z)$ is then defined by

$$Y_*^{\sigma,(l)}(v, z) = W_*\big(\exp(\Delta_{z*})v, z\big) \quad \text{for } v \in V_L. \tag{4.3.9}$$

Summarizing, we get a well-defined linear map (the relative σ-twisted vertex operator map)

$$\begin{aligned} V_L &\to \big(\operatorname{End} V_L^T\big)\{z\} \\ v &\mapsto Y_*^{\sigma,(l)}(v, z) = \sum_{n \in \mathbf{C}} v_n z^{-n-1}, \quad v_n \in \operatorname{End} V_L^T, \end{aligned} \tag{4.3.10}$$

which satisfies $Y_*^{\sigma,(l)}(v, z) \in (\operatorname{End} V_L^T)[\![z^{1/6}, z^{-1/6}]\!]$ if $\mathfrak{h}_* = 0$. We refer to [15] for a discussion of the properties of these operators. Of particular importance for our purposes is the fact that

$$\big[\hat{\mathfrak{h}}_*[\sigma], Y_*^{\sigma,(l)}(v, z)\big] = 0 \quad \text{for } v \in V_L \tag{4.3.11}$$

(cf. [15, Proposition 4.6]), where $\hat{\mathfrak{h}}_*[\sigma]$ is defined as in (4.3.1)–(4.3.2).

By using a single copy of Q and the space $\mathfrak{h}_* = 0$ we get a well-defined linear map

$$\begin{aligned} V_Q &\to \left(\operatorname{End} V_Q^T\right)[\![z^{1/6}, z^{-1/6}]\!] \\ v &\mapsto Y^{\sigma,(1)}(v, z) = \sum_{n \in \frac{1}{6}\mathbf{Z}} v_n z^{-n-1}. \end{aligned}$$

Let $\hat{\mathfrak{s}}_i[\sigma] = \bigoplus_{n \in \frac{1}{6}\mathbf{Z}} \mathfrak{s}_{i_{(6n)}} \otimes t^n \oplus \mathbf{C}c$ and $\hat{\mathfrak{s}}_i[\sigma]_\pm = \bigoplus_{n \in \frac{1}{6}\mathbf{Z},\, \pm n > 0} \mathfrak{s}_{i_{(6n)}} \otimes t^n$, $1 \leqslant i \leqslant l$. Since $S(\hat{\mathfrak{h}}[\sigma]_-) = S(\tilde{\mathfrak{h}}[\sigma]'_-) = \bigotimes_{i=1}^l S(\hat{\mathfrak{s}}_i[\sigma]_-)$, (4.3.2) implies that

$$V_L^T \cong \bigotimes_{i=1}^{l} V_{Q_i}^T \quad \text{(linearly).} \tag{4.3.12}$$

Then from [36, Section 9] and [3, Sections 8–9] (see also [21, Section 7.4, 37]) one gets the following theorem.

Theorem 4.3.1. *The linear map $\rho^\sigma : \hat{\mathfrak{g}}[\sigma] \to \operatorname{End} V_Q^T$ given by*

$$\rho^\sigma(c) = 1, \qquad \rho^\sigma\big(x(\sigma; z)\big) = Y^{\sigma,(1)}\big(i(x), z\big), \quad x \in \mathfrak{g}, \tag{4.3.13}$$

defines a level 1 *$\hat{\mathfrak{g}}[\sigma]$-module structure on V_Q^T such that $V_Q^T \cong L(\Lambda_1; \hat{\mathfrak{g}}[\sigma])$. Moreover, the linear map $\pi^\sigma : \hat{\mathfrak{g}}[\sigma] \to \operatorname{End} V_L^T$ determined by*

$$\pi^\sigma(c) = l, \qquad \pi^\sigma\big(x(\sigma, z)\big) = Y^{\sigma,(l)}\big(i(x), z\big), \quad x \in \mathfrak{g}, \tag{4.3.14}$$

defines a level l $\hat{\mathfrak{g}}[\sigma]$-module structure on V_L^T, and this representation is isomorphic to the tensor product of the representations

$$\pi_j^\sigma : \hat{\mathfrak{g}}[\sigma] \to \operatorname{End} V_{Q_j}^T, \quad 1 \leqslant j \leqslant l, \tag{4.3.15}$$

given by $\pi_j^\sigma(c) = 1$, $\pi_j^\sigma(x(\sigma; z)) = Y^{\sigma,(1)}(i_j(x), z)$, $x \in \mathfrak{g}$.

Remark 4.3.2. Under the isomorphism in the second part of Theorem 4.3.1 the action of the operator $Y^{\sigma,(1)}(i_j(x), z)$ on V_L^T coincides with the action of $1 \otimes \cdots \otimes 1 \otimes Y^{\sigma,(1)}(i_j(x), z) \otimes 1 \otimes \cdots \otimes 1$. Moreover, the representations (4.3.13)–(4.3.15) extend to $\tilde{\mathfrak{g}}[\sigma]$ by letting d act as the degree operator.

Corollary 4.3.3. *On the subspace $L(l\Lambda_0; \hat{\mathfrak{g}})$ of V_L the map $Y^{\sigma,(l)}(\cdot, z)$ of* (4.3.14) *coincides with the map $Y^\sigma(\cdot, z)$ of Theorems* 3.1.5 *and* 3.2.5.

Proof. By [15, Theorem 7.1], $(V_L, Y^{(l)}, \mathbf{1}, \omega)$ is a VOA and $(V_L^T, Y^{\sigma,(l)})$ is a σ-twisted V_L-module (see [9,15] for the construction of the Virasoro element $\omega \in V_L$). In particular, the map $Y^{\sigma,(l)}$ satisfies the twisted associator formula (3.1.2) with $Y_M = Y^{\sigma,(l)}$ and $Y = Y^{(l)}$. Let M be any standard level l $\hat{\mathfrak{g}}[\sigma]$-module. According to Section 2.1, (4.3.12), and Theorem 4.3.1, M can be isomorphically embedded into $L(\Lambda_1;\hat{\mathfrak{g}}[\sigma])^{\otimes l}\cong V_L^T$. Denoting by $x_{(6n)}(n)$ the operator $\pi^\sigma(x_{(6n)}\otimes t^n)$ on V_L^T, $x\in\mathfrak{g}$, $n\in\frac{1}{6}\mathbf{Z}$, and comparing the expression for $x(\sigma;z)$ from (4.1.11) with the generic expansion $Y^{\sigma,(l)}(i(x),z)=\sum_{n\in\frac{1}{6}\mathbf{Z}}x_n z^{-n-1}$ (cf. (4.3.10)), we see that $x_n=x_{(6n)}(n)$. For $x\in\mathfrak{g}$ one has therefore that $Y^{\sigma,(l)}(i(x),z)=Y^\sigma(x(-1)\mathbf{1},z)$ when acting on M, where (as before) $\mathbf{1}=1\otimes\iota(1)\in L(l\Lambda_0;\hat{\mathfrak{g}})$ and $Y^\sigma(\cdot\,,z)$ is the map defined in (3.1.5). Then the corollary follows from the twisted iterate formula (3.1.2) and Corollary 4.2.3. $\square$

4.4. The GVOA Ω^A_{2k+1} and its action on the vacuum spaces $\Omega_{k_0,2k+1}$

Set now

$$\begin{aligned}&\Omega_*=\bigl\{v\in V_L\ \big|\ h(n)v=0 \text{ for } h\in\mathfrak{h}_*,\ n>0\bigr\}\subset V_L,\\&\Omega_*^\sigma=\bigl\{v\in V_L^T\ \big|\ h(n)v=0 \text{ for } h\in\mathfrak{h}_*,\ n\in\tfrac{1}{6}\mathbf{Z}_+\bigr\}\subset V_L^T.\end{aligned}\tag{4.4.1}$$

Clearly, Ω_* is the vacuum space for the action of the Heisenberg algebra $\tilde{\mathfrak{h}}'_*$ on V_L, while Ω_*^σ is the vacuum space for the action of the Heisenberg algebra $\hat{\mathfrak{h}}_*[\sigma]$ on V_L^T. By (4.2.11) and (4.3.11), these spaces are preserved by the operators $Y_*^{(l)}(v,z)$, $v\in V_L$, and $Y_*^{\sigma,(l)}(v,z)$, $v\in V_L$, respectively. Since $S(\tilde{\mathfrak{h}}'_-)=S((\tilde{\mathfrak{h}}_*)'_-)\otimes S((\tilde{\mathfrak{h}}_*^\perp)'_-)$, one gets from (4.2.6) that

$$\Omega_*=S\bigl(\bigl(\tilde{\mathfrak{h}}_*^\perp\bigr)'_-\bigr)\otimes\mathbf{C}\{L\},\tag{4.4.2}$$

and thus V_L may be decomposed as

$$\begin{aligned}&V_L=\Omega_*\oplus V_*,\\&V_*=\mathbf{C}\text{-span}\bigl\{h(n)V_L\ \big|\ h\in\mathfrak{h}_*,\ n<0\bigr\}=\bigl(\tilde{\mathfrak{h}}_*\bigr)'_-S\bigl(\bigl(\tilde{\mathfrak{h}}_*\bigr)'_-\bigr)\otimes\Omega_*.\end{aligned}\tag{4.4.3}$$

Note that by construction the operators $Y_*^{\sigma,(l)}$ satisfy

$$Y_*^{\sigma,(l)}(v,z)=0\quad\text{ if } v\in V_*.\tag{4.4.4}$$

In similar fashion, (4.3.2) yields $\Omega_*^\sigma=S(\hat{\mathfrak{h}}_*^\perp[\sigma]_-)\otimes T$.

Recall (4.1.2) and (4.1.12) and consider the diagonal embedding

$$\begin{aligned}{}_*:\mathfrak{s}&\to\mathfrak{h}\\\alpha&\mapsto\alpha_*=\alpha_1+\cdots+\alpha_l,\end{aligned}\tag{4.4.5}$$

where $\alpha_i\in Q_i\otimes_{\mathbf{Z}}\mathbf{C}$ corresponds to $\alpha\in\mathfrak{s}$. Obviously, the map $_*$ defines a group isomorphism between Q and Q_*. We assume henceforth that the subspace $\mathfrak{h}_*$ of $\mathfrak{h}$ (see (4.1.13)) is

$$\mathfrak{h}_*=\{\alpha_*\mid\alpha\in\mathfrak{s}\}=\mathfrak{s}_*.\tag{4.4.6}$$

Then $\mathfrak{h}_*^\perp=\mathbf{C}\text{-span}\{\alpha_i-\alpha_j\mid\alpha\in\mathfrak{s},\ i,j=1,\dots,l\}$ (cf. (4.1.13)). Define

$$A=\{e_{\alpha_*}\mid\alpha\in 2Q\}\subset\widehat{L}.\tag{4.4.7}$$

By the choice of the cocycle ε_0 and of the section e (cf. (4.1.4)–(4.1.5) and (4.1.10)), A is a central subgroup of $\widehat{L}$ isomorphic to the subgroup $2Q_*$ of Q_* and such that $A \cap \langle \eta \rangle = 1\ (= e_0)$. Moreover, $\langle L, \overline{A} \rangle \in 2\mathbf{Z}$, $\mathfrak{h}_* \cap L$ $\mathbf{C}$-spans $\mathfrak{h}_*$, and $\operatorname{rank} A = \operatorname{rank} \mathfrak{g} = \dim \mathfrak{s} = 2$, so that $\overline{A}$ $\mathbf{C}$-spans $\mathfrak{h}_*$. Recall the projections (4.1.15) and notice that

$$\alpha_i'' = l^{-1}\alpha_* \quad \text{for } \alpha \in \Phi,\ i = 1, \ldots, l, \tag{4.4.8}$$

so that

$$L'' = \{\beta'' \mid \beta \in L\} = l^{-1}Q_*. \tag{4.4.9}$$

Let $G = L''/\overline{A} = l^{-1}Q_*/2Q_* \cong l^{-1}Q/2Q$ and notice that the smallest positive integer S such that $\langle \alpha', \beta' \rangle \in S^{-1}\mathbf{Z}$, $\langle \alpha, \beta \rangle \in S^{-1}\mathbf{Z}$, and $\langle \alpha', \alpha' \rangle \in 2S^{-1}\mathbf{Z}$ for $\alpha, \beta \in L$ is precisely

$$S = l. \tag{4.4.10}$$

A well-defined symmetric nondegenerate $l^{-1}\mathbf{Z}/2\mathbf{Z}$-valued $\mathbf{Z}$-bilinear form on G is then given by

$$\begin{aligned} G \times G &\to l^{-1}\mathbf{Z}/2\mathbf{Z} \\ \bigl(l^{-1}\alpha_* + \overline{A}, l^{-1}\beta_* + \overline{A}\bigr) &\mapsto l^{-1}\langle \alpha, \beta \rangle + 2\mathbf{Z} \quad \text{for } \alpha, \beta \in Q. \end{aligned}$$

In the remainder of this section we shall denote the standard $\hat{\mathfrak{g}}$*-module* $L(l\Lambda_0; \hat{\mathfrak{g}})$ *simply by* $L(l, 0)$. Recall that $L(l, 0) \cong U(\hat{\mathfrak{g}}) \cdot \iota(1) \subset V_L$ (cf. Corollary 4.2.3). The action of $\hat{\mathfrak{g}}$ on V_L being given by Theorem 4.2.1, one gets from [9, Proposition 14.5] and the complete reducibility of V_L as a $\hat{\mathfrak{g}}$-module that any $\hat{\mathfrak{g}}$-submodule M of V_L is A-stable and has an $\hat{\mathfrak{h}}_*$-stable complement in V_L. In particular, this holds for $M = L(l, 0)$. Define the following spaces:

$$\Omega_{L(l,0)} = \Omega_* \cap L(l, 0), \tag{4.4.11}$$

$$W_{L(l,0)} = \mathbf{C}\text{-span}\bigl\{v - a \cdot v \bigm| v \in L(l, 0),\ a \in A\bigr\} \subset L(l, 0), \tag{4.4.12}$$

$$\Omega^A_{L(l,0)} = \Omega_{L(l,0)}/(\Omega_* \cap W_{L(l,0)}) = (\Omega_{L(l,0)} + W_{L(l,0)})/W_{L(l,0)}. \tag{4.4.13}$$

Note that by (3.2.22), (4.4.1), and (4.4.11), one has in fact that $\Omega_{L(l,0)} = \Omega_{2k+1}$. Since $L(l, 0)$ has an A-stable and $\hat{\mathfrak{h}}_*$-stable complement in V_L, it follows that $\tilde{\mathfrak{h}}'_*$ acts naturally on $L(l, 0)/W_{L(l,0)}$, so that $\Omega^A_{L(l,0)}$ may be described alternatively as

$$\begin{aligned} \Omega^A_{L(l,0)} &= \bigl\{v \in L(l, 0)/W_{L(l,0)} \bigm| h(n)v = 0 \text{ for } h \in \mathfrak{h}_*,\ n > 0\bigr\} \\ &= \bigl\{v \in L(l, 0)/W_{L(l,0)} \bigm| h(n)v = 0 \text{ for } h \in \mathfrak{s},\ n > 0\bigr\} \subset \Omega^A_*, \end{aligned}$$

where for $h \in \mathfrak{s}$, $h(n)$ denotes the operator $\pi(h \otimes t^n) = h_*(n)$ of (4.2.15) and

$$\Omega^A_* = \bigl\{v \in V_L/W_{V_L} \bigm| h(n)v = 0 \text{ for } h \in \mathfrak{h}_*,\ n > 0\bigr\},$$

with $W_{V_L} = \mathbf{C}\text{-span}\{v - a \cdot v \mid v \in V_L,\ a \in A\} \subset V_L$. One may then define a G-gradation on $\Omega^A_{L(l,0)}$ in the following way: note first that by (4.4.2) and (4.4.8)–(4.4.9) Ω_* decomposes as

$$\Omega_* = \coprod_{\lambda \in l^{-1}Q} \Omega_*^{\lambda_*} \quad \text{with} \quad \Omega_*^{\lambda_*} = \left\{v \in \Omega_* \;\middle|\; h(0)v = \langle h_*, \lambda_* \rangle v,\ h \in \mathfrak{s}\right\}$$
$$= \sum_{a \in \widehat{L},\, \bar{a}'' = \lambda_*} S\big(\big(\tilde{\mathfrak{h}}_*^{\perp}\big)'_-\big) \otimes \iota(a). \tag{4.4.14}$$

Since $\Omega_{L(l,0)}$ is stable under the operators $h(0)$, $h \in \mathfrak{s}$, we see that $\Omega_{L(l,0)} = \coprod_{\lambda \in l^{-1}Q} \Omega^{\lambda_*}_{L(l,0)}$, where $\Omega^{\lambda_*}_{L(l,0)} = \Omega_{L(l,0)} \cap \Omega^{\lambda_*}_* = L(l,0) \cap \Omega^{\lambda_*}_*$. It then follows from (4.4.13) that $\Omega^A_{L(l,0)}$ is G-graded, with

$$\big(\Omega^A_{L(l,0)}\big)^g = \sum_{\lambda \in l^{-1}Q,\, \lambda_* + 2Q_* = g} \big(\Omega^{\lambda_*}_{L(l,0)} + W_{L(l,0)}\big)/W_{L(l,0)} \quad \text{for } g \in G.$$

Note that by (4.4.7) and (4.4.11)–(4.4.12) one has that if $\lambda, \mu \in l^{-1}Q$ are such that $\lambda_* - \mu_* \in 2Q_*$ then $(\Omega^{\lambda_*}_{L(l,0)} + W_{L(l,0)})/W_{L(l,0)} = (\Omega^{\mu_*}_{L(l,0)} + W_{L(l,0)})/W_{L(l,0)}$, so that in particular

$$\big(\Omega^A_{L(l,0)}\big)^0 = \big(\Omega^0_{L(l,0)} + W_{L(l,0)}\big)/W_{L(l,0)}. \tag{4.4.15}$$

Recall now the space Ω^0_l from (3.2.23) (with $l = 2k+1$) and the element $\omega_2 = \omega' - \omega_1 \in \Omega^0_l$ (cf. (3.2.20)–(3.2.21) and (3.2.24)). Then clearly

$$\Omega^0_l = \Omega^0_{L(l,0)}. \tag{4.4.16}$$

Moreover, by using the actions of Theorem 4.2.1, one can rewrite ω', ω_1, and ω_2 respectively as

$$\omega' = \frac{1}{2(l+3)} \sum_{i=1}^{8} a_i(-1)a_{\pi(i)}(-1) \cdot \iota(1),$$
$$\omega_1 = \frac{1}{2} \sum_{i=1}^{2} \gamma^i(-1)\gamma^{\pi(i)}(-1) \cdot \iota(1), \qquad \omega_2 = \omega' - \omega_1, \tag{4.4.17}$$

where $\gamma^i = (a_i/\sqrt{l})_* \in \mathfrak{h}_*$ for $i = 1, 2$ (so that $\{\gamma^1, \gamma^2\}$ and $\{\gamma^2, \gamma^1\}$ are dual bases of $\mathfrak{h}_*$) and $\mathbf{1}$ is identified with $1 \otimes \iota(1)$. Then ω_2 becomes exactly the element $\omega_{\mathcal{G}_l, \mathcal{H}_l}$ of [9, (14.53)].

We shall need the following lemma.

Lemma 4.4.1. *With the above notations, one has* $\Omega^0_{L(l,0)} \cap W_{L(l,0)} = \{0\}$.

Proof. Recall the group A defined in (4.4.7) and the action of $\widehat{L}$ on $\mathbf{C}\{L\}$ given by (4.2.1), and let $U = \Omega^0_{L(l,0)} \cap W_{L(l,0)}$ and $v \in U$. Since $U \subset \Omega_{L(l,0)} \cap W_{L(l,0)}$, it follows from (4.4.2), (4.4.11), and (4.4.12) that there exist positive integers n, m_i $(1 \leqslant i \leqslant n)$, such that

$$v = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \bigl[u_{ij} \otimes \iota(b_{ij}) - a_i \cdot \bigl(u_{ij} \otimes \iota(b_{ij})\bigr)\bigr], \tag{4.4.18}$$

where $u_{ij} \in S((\tilde{\mathfrak{h}}_*^{\perp})'_-)$, $b_{ij} \in \widehat{L}$, and $a_i \in A$. Recall also the linear maps of (4.2.7), (4.2.13), and (4.4.5) and notice that for $h \in \mathfrak{s}$, $n \in \mathbf{Z}$, the operator $h(n)$ acts on v as the "unrelativized" operator $\pi(h \otimes t^n) = h_*(n)$ of (4.2.15). Since $a_i \cdot (u_{ij} \otimes \iota(b_{ij})) = u_{ij} \otimes \iota(a_i b_{ij})$, one gets in particular

$$\begin{aligned} &h(0)\bigl(u_{ij} \otimes \iota(b_{ij})\bigr) = \langle h_*, \bar{b}''_{ij}\rangle u_{ij} \otimes \iota(b_{ij}), \\ &h(0)\bigl[a_i \cdot \bigl(u_{ij} \otimes \iota(b_{ij})\bigr)\bigr] = \langle h_*, \bar{b}''_{ij} + \bar{a}''_i\rangle a_i \cdot \bigl(u_{ij} \otimes \iota(b_{ij})\bigr) \end{aligned} \tag{4.4.19}$$

for $h \in \mathfrak{s}$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$. In terms of the gradation (4.4.14), (4.4.19) means that the corresponding vectors have degrees $\bar{b}''_{ij}$ and $\bar{b}''_{ij} + \alpha^i_*$, respectively, where $\alpha^i \in 2Q$ are such that $a_i = e_{\alpha^i_*}$, $1 \leqslant i \leqslant n$. Denote by $\lambda_1, \ldots, \lambda_m \in \frac{1}{l} Q_*$ the distinct degrees occurring among the vectors in the right-hand side of (4.4.18). Then one may rewrite (4.4.18) as

$$v = v_1 + \cdots + v_m \tag{4.4.20}$$

with $v_i \in \Omega_*$ of degree λ_i. If $m = 1$ one gets that $\bar{b}''_{ij} = \bar{b}''_{ij} + \alpha^i_* = \lambda_1$ for all i and j, hence $\alpha^i_* = 0$ and $a_i = e_0 = 1$, $1 \leqslant i \leqslant n$, by the choice of the section e (cf. (4.1.10)). Then (4.4.18) implies that $v = 0$ in this case. Assume now that $m \geqslant 2$ and choose $h \in \mathfrak{s}$ such that the scalars $\langle h_*, \lambda_i\rangle$ are distinct (such h exists since the λ_i's are distinct linear forms on $\mathfrak{h}_*$). Then using (4.4.20) and the fact that $v \in \Omega^0_{L(l,0)}$ one obtains $v_1 + \cdots + v_m = v$ and $\sum_{k=1}^{m} \langle h_*, \lambda_k\rangle^j v_k = h(0)^j v = 0$ for $1 \leqslant j \leqslant m-1$. This is by construction a system of $m \times m$ linear equations with nonsingular matrix, so that $v_k \in \mathbf{C}v \subset U$ for $1 \leqslant k \leqslant m$. In particular $v_i \in \Omega^0_{L(l,0)}$ and thus $\lambda_i = 0$ for all i. Therefore, $\bar{b}''_{ij} = \bar{b}''_{ij} + \alpha^i_* = 0$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m_i$, so that $\alpha^i_* = 0$, $1 \leqslant i \leqslant n$, and consequently $a_i = e_{\alpha^i_*} = 1$, $1 \leqslant i \leqslant n$. Then (4.4.18) again implies that $v = 0$, which completes the proof. □

Lemma 4.4.1 and (4.4.15)–(4.4.16) yield a linear isomorphism

$$\Omega^0_l \xrightarrow{\cong} \bigl(\Omega^0_{L(l,0)} + W_{L(l,0)}\bigr)/W_{L(l,0)} = \bigl(\Omega^A_{L(l,0)}\bigr)^0 \tag{4.4.21}$$

by means of which we identify Ω^0_l and $(\Omega^A_{L(l,0)})^0$. It follows from (4.2.9) that $Y^{(l)}_*(a \cdot v, z) = aY^{(l)}_*(v, z)$ for $v \in V_L$, $a \in A$. (This is true even for $a \in \widehat{L}$ such that $\bar{a} \in L \cap \mathfrak{h}_*$.) Therefore, the vertex map $Y^{(l)}_*$ of (4.2.10) induces a well-defined linear map

$$\begin{aligned}\overline{Y}_*^{(l)}(\cdot\,,z):\Omega^A_{L(l,0)} &\to \left(\operatorname{End}\Omega^A_{L(l,0)}\right)[\![z^{1/l},z^{-1/l}]\!]\\ v+W_{L(l,0)} &\mapsto \overline{Y}_*^{(l)}(v+W_{L(l,0)},z)=Y_*^{(l)}(v,z)\\ &= \sum_{n\in\frac{1}{l}\mathbf{Z}}(v+W_{L(l,0)})_n z^{-n-1},\end{aligned}\tag{4.4.22}$$

which is called the quotient relative untwisted vertex operator map defined with respect to the group A (cf. [9, Chapters 4 and 14]).

Lemma 4.4.2. *For $v\in\Omega_l^0$ one has $\overline{Y}_*^{(l)}(v,z)=Y^{(l)}(v,z)=Y(v,z)$ as (series of) operators acting on $\Omega^A_{L(l,0)}$.*

Proof. Let $v\in\Omega_l^0$. By (4.4.14) one may write $v=\sum_{i=1}^k v_i\otimes\iota(a_i)$ for some $k\in\mathbf{Z}_+$, $0\neq v_i\in S((\tilde{\mathfrak{h}}_*^{\perp})'_-)$, and $a_i\in\widehat{L}$, $1\leqslant i\leqslant k$, such that the $\iota(a_i)$'s are linearly independent. Then the relations $0=h(0)v=\sum_{i=1}^k\langle h_*,\bar{a}_i\rangle v_i\otimes\iota(a_i)$, $h\in\mathfrak{s}$, imply that $\langle h_*,\bar{a}_i\rangle=0$, $h\in\mathfrak{s}$, $1\leqslant i\leqslant k$. This means that $\bar{a}_i''=0$ or, equivalently, $\bar{a}_i'=\bar{a}_i$, $1\leqslant i\leqslant k$. It then follows from (4.2.8) and (4.2.9) that $Y_*^{(l)}(v,z)=Y^{(l)}(v,z)$, which combined with (4.4.21), (4.4.22), and Corollary 4.2.3 proves the lemma. □

Recall that $l=2k+1$ and that $\operatorname{rank}\Omega_l^0=c_2(k)$. By using the results of [9, Chapter 14] together with Theorem 4.2.1, Corollary 4.2.3, (4.4.21), and Lemma 4.4.2, we get:

Theorem 4.4.3. *Let $\overline{Y}_*^{(l)}(\omega_2+W_{L(l,0)},z)=\sum_{n\in\mathbf{Z}}\overline{L}_*(n)z^{-n-2}$. Then the structure*

$$\left(\Omega^A_{L(l,0)},\overline{Y}_*^{(l)},\ \iota(1)+W_{L(l,0)},\ \omega_2+W_{L(l,0)},\ l,G,(\cdot\,,\cdot)\right)$$

is a simple GVOA of rank $c_2(k)$ generated by the set $\{i(x_\alpha)+W_{L(l,0)}\mid\alpha\in\Phi\}$ that consists of $\overline{L}_*(0)$-eigenvectors with eigenvalue $1-l^{-1}$. Moreover, by restricting the map $\overline{Y}_*^{(l)}$ to the subspace $(\Omega^A_{L(l,0)})^0\equiv\Omega_l^0$, the structure $(\Omega_l^0,\overline{Y}_*^{(l)},\iota(1)+W_{L(l,0)},\omega_2+W_{L(l,0)})$ becomes a simple VOA of rank $c_2(k)$ such that $\overline{Y}_*^{(l)}(v,z)=Y(v,z)$ for every $v\in\Omega_l^0$.

Note that by (4.4.12), one can actually rewrite $\iota(1)+W_{L(l,0)}$ and $\omega_2+W_{L(l,0)}$ as $\iota(A)$ and $\omega_2\otimes\iota(A)$, respectively, where ω_2 is identified with $\omega_2\otimes\iota(1)$.

We now focus on the operators $Y_*^{\sigma,(l)}(\cdot\,,z)$ introduced in Section 4.3. We show first the following lemma.

Lemma 4.4.4. *Let $a\in A$ and $v\in V_L$. Then $Y_*^{\sigma,(l)}(a\cdot v,z)=Y_*^{\sigma,(l)}(v,z)$ as (series of) operators acting on V_L^T.*

Proof. Since $Y_*^{\sigma,(l)}(\cdot\,,z)$ is linear, it suffices to prove the lemma for $v \in V_L$ of the form $v = v^* \otimes \iota(b)$, where $v^* \in S(\tilde{\mathfrak{h}}'_-)$ and $b \in \widehat{L}$. Recall (4.1.4) and (4.1.10). It follows from [15, Proposition 4.6] that if $a \in A$ (or more generally, if $a \in \widehat{L}$ is such that $\bar{a} \in L \cap \mathfrak{h}_*$) and $v \in V_L$, then $Y_*^{\sigma,(l)}(a \cdot v, z) = \varepsilon(\bar{a}, \bar{b}) a Y_*^{\sigma,(l)}(v, z)$, where in the right-hand side a is understood to act on T via the set-theoretic identification between $\widehat{L}$ and $\widehat{L}_\sigma$ given by (4.1.8) (see also (4.3.6)). Thus $Y_*^{\sigma,(l)}(a \cdot v, z) = aY_*^{\sigma,(l)}(v, z)$ by (4.4.7) and (4.1.4), and it is therefore enough to show that a acts as 1 on T in order to complete the proof. Using (4.4.7) and the fact that $\varepsilon(2\alpha, \beta) = 1$ for $\alpha, \beta \in L$, we see that the set-theoretic identification (4.1.8) restricts to an actual group isomorphism on A, so that T becomes an A-module. But $\widehat{L}_\sigma = \langle \eta \rangle \times L$ and L acts as 1 on T, hence $(1, \alpha)$ acts as 1 on T for every $\alpha \in L$. In particular, T must be a trivial A-module, as needed. $\square$

It follows from Lemma 4.4.4, (4.4.10), and the definition of relative σ-twisted vertex operators in Section 4.3 that $Y_*^{\sigma,(l)}(\cdot\,,z)$ induces a well-defined quotient linear map

$$\begin{aligned} \overline{Y}_*^{\sigma,(l)}(\cdot\,,z) : \Omega^A_{L(l,0)} &\to \big(\operatorname{End} V_L^T\big)\big[\!\big[z^{1/6l}, z^{-1/6l}\big]\!\big] \\ v + W_{L(l,0)} &\mapsto \overline{Y}_*^{\sigma,(l)}(v + W_{L(l,0)}, z) = Y_*^{\sigma,(l)}(v, z), \end{aligned} \tag{4.4.23}$$

where $Y_*^{\sigma,(l)}(\cdot\,,z)$ denotes the restriction to $\Omega_{L(l,0)}$ of the map $Y_*^{\sigma,(l)}(\cdot\,,z)$ of (4.3.10). We may therefore define the following subalgebra of $\operatorname{End} V_L^T$:

$$\begin{aligned} \overline{\mathcal{Y}}_{V_L^T} &= \mathbf{C}\text{-span}\big\{\operatorname{Res}_z z^n \overline{Y}_*^{\sigma,(l)}(v + W_{L(l,0)}, z) \;\big|\; v \in \Omega^A_{L(l,0)},\ n \in \tfrac{1}{6l}\mathbf{Z}\big\} \\ &= \mathbf{C}\text{-span}\big\{\operatorname{Res}_z z^n Y_*^{\sigma,(l)}(v, z) \;\big|\; v \in \Omega_{L(l,0)},\ n \in \tfrac{1}{6l}\mathbf{Z}\big\}. \end{aligned} \tag{4.4.24}$$

Let now $L(\Lambda; \hat{\mathfrak{g}}[\sigma])$ be any level l standard $\mathfrak{g}[\sigma]$-module. Then $\Lambda = k_0\Lambda_0 + (l - 2k_0)\Lambda_1$ for some $k_0 \in \{0, 1, \ldots, k\}$ by (2.2.4), so that $L(\Lambda; \hat{\mathfrak{g}}[\sigma]) = L_{k_0,l}(\hat{\mathfrak{g}}[\sigma])$ in the notation of Section 2.1. According to the proof of Corollary 4.3.3, $L_{k_0,l}(\hat{\mathfrak{g}}[\sigma])$ can be isomorphically embedded into $L(\Lambda_1; \hat{\mathfrak{g}}[\sigma])^{\otimes l} \cong V_L^T$. Furthermore, Remark 3.2.3 and (4.4.6) imply that the principal Heisenberg subalgebra $\tilde{\mathfrak{s}}[\sigma]'$ $(= \hat{\mathfrak{s}}[\sigma])$ of $\tilde{\mathfrak{g}}[\sigma]$ may be identified with $\hat{\mathfrak{h}}_*[\sigma]$ $(\cong \hat{\mathfrak{s}}_*[\sigma])$. Consequently, (2.2.10) may be rewritten as

$$L_{k_0,l}\big(\hat{\mathfrak{g}}[\sigma]\big) = S\big(\hat{\mathfrak{h}}_*[\sigma]_-\big) \otimes \Omega_{k_0,l}, \tag{4.4.25}$$

with $\Omega_{k_0,l} \in \mathcal{O}_2(k)$, the corresponding vacuum space (see (2.2.10) and (2.2.12)). Note also that (4.3.14) and (4.4.1) actually imply that

$$\Omega_{k_0,l} = L_{k_0,l}\big(\hat{\mathfrak{g}}[\sigma]\big) \cap \Omega_*^\sigma. \tag{4.4.26}$$

For $\phi \in \mathfrak{h}$ let us define the following operators:

$$E^{\pm}_{(l)}(\phi, z) = \exp\Bigg[\sum_{n \in \frac{1}{6}\mathbf{Z},\, \pm n > 0} \frac{\phi_{(6n)} \otimes t^n}{ln} z^{-n}\Bigg] = E^{\pm}_{(1)}(\phi/l, z), \tag{4.4.27}$$

where $E^{\pm}_{(1)}(\cdot\,,z)$ is the unrelativized version of the operator $E^{\pm}_{*(1)}(\cdot\,,z)$ defined in (4.3.5).

Lemma 4.4.5. *The component operators of $Y_*^{\sigma,(l)}(v,z)$, $v\in L(l,0)$, preserve each of the standard $\hat{\mathfrak{g}}[\sigma]$-modules $L_{k_0,l}(\hat{\mathfrak{g}}[\sigma])$, $k_0\in\{0,1,\ldots,k\}$.*

Proof. Let $k_0\in\{0,1,\ldots,k\}$ and set $M=L_{k_0,l}(\hat{\mathfrak{g}}[\sigma])$. By (4.4.3)–(4.4.4) and (4.4.11) it suffices to show that $Y_*^{\sigma,(l)}(v,z)$ preserves M for any $v\in\Omega_{L(l,0)}$. Recall from Theorem 4.4.3 that

$$\Omega_{L(l,0)}=\mathbf{C}\text{-span}\Big\{v^{(1)}_{n_1}\cdots v^{(j)}_{n_j}\cdot\iota(1)\;\Big|\;v^{(i)}\in\{i(x_\alpha)\mid\alpha\in\Phi\}\cup\{\iota(1)\},\\ n_i\in\tfrac{1}{l}\mathbf{Z},\ i=1,\ldots,j\Big\},$$

where $v^{(i)}_{n_i}$ is understood as a component operator of $Y_*^{(l)}(v^{(i)},z)$. Thus it is enough to prove that M is invariant under $Y_*^{\sigma,(l)}(v,z)$ for $v\in\Omega_{L(l,0)}$ of the form

$$v=v^{(1)}_{n_1}\cdots v^{(j)}_{n_j}\cdot\iota(1),\quad \text{with } v^{(i)}\in\big\{i(x_\alpha)\mid\alpha\in\Phi\big\},\ n_i\in\tfrac{1}{l}\mathbf{Z},\ i=1,\ldots,j. \tag{4.4.28}$$

Let therefore $\alpha^m\in\Phi$ be such that $v^{(m)}=i(x_{\alpha^m})$, $m=1,\ldots,j$, and set $\alpha=\sum_{m=1}^j\alpha^m\in Q$. It follows from (4.2.1) and (4.2.8)–(4.2.9) (see also (4.4.2)) that one may write v as

$$v=\sum_{r_1,\ldots,r_j=1}^{l}u_{r_1,\ldots,r_j}, \tag{4.4.29}$$

where for any $r_1,\ldots,r_j\in\{1,2,\ldots,l\}$ one has

$$u_{r_1,\ldots,r_j}=v_{r_1,\ldots,r_j}\otimes\iota(e_{r_1,\ldots,r_j}),\quad \text{with}\\ v_{r_1,\ldots,r_j}\in S\big((\tilde{\mathfrak{h}}_*^{\perp})'_-\big),\ e_{r_1,\ldots,r_j}=e_{\alpha^1_{r_1}+\cdots+\alpha^j_{r_j}},$$

and $\alpha^m_{r_m}$ denotes the r_mth copy of $\alpha^m\in\Phi$ in L. Then (4.4.29) and (4.3.8) imply that

$$\exp(\Delta_{z*})u_{r_1,\ldots,r_j}=\exp(\Delta_z)u_{r_1,\ldots,r_j},$$

which combined with (4.3.9) and (4.3.7) gives

$$\begin{aligned}Y_*^{\sigma,(l)}(u_{r_1,\ldots,r_j},z)&=W_*\big(\exp(\Delta_{z*})u_{r_1,\ldots,r_j},z\big)\\&=W_*\big(\exp(\Delta_z)u_{r_1,\ldots,r_j},z\big).\end{aligned} \tag{4.4.30}$$

Note next that for any $r_1,\ldots,r_j\in\{1,2,\ldots,l\}$ and $p\in\{0,1,\ldots,5\}$ one has by (4.4.8) that

$$\sigma^p\big(\alpha^1_{r_1}+\cdots+\alpha^j_{r_j}\big)'=\sigma^p\big(\alpha^1_{r_1}+\cdots+\alpha^j_{r_j}\big)-l^{-1}\sigma^p(\alpha_*),$$

and thus

$$\begin{aligned}&\big\langle\sigma^p\big(\alpha^1_{r_1}+\cdots+\alpha^j_{r_j}\big)',\big(\alpha^1_{s_1}+\cdots+\alpha^j_{s_j}\big)'\big\rangle\\&\quad=\big\langle\sigma^p\big(\alpha^1_{r_1}+\cdots+\alpha^j_{r_j}\big),\big(\alpha^1_{s_1}+\cdots+\alpha^j_{s_j}\big)\big\rangle-l^{-1}\big\langle\sigma^p(\alpha),\alpha\big\rangle\end{aligned}\tag{4.4.31}$$

for all $r_1,\dots,r_j,s_1,\dots,s_j\in\{1,2,\dots,l\}$ and $p\in\{0,1,\dots,5\}$. Recall (4.3.4) and let

$$C_1(\alpha)=\tau\Big(\frac{\alpha_*}{l}\Big)^{-1},\qquad C_2(\alpha)=\frac{\langle\alpha,\alpha\rangle}{2l},\qquad C_3(\alpha)=6^{C_2(\alpha)}C_1(\alpha).$$

It is then clear from (4.3.4) and (4.4.31) that for any $r_1,\dots,r_j\in\{1,2,\dots,l\}$,

$$\tau\big(\big(\alpha^1_{r_1}+\cdots+\alpha^j_{r_j}\big)'\big)=C_1(\alpha)\tau\big(\alpha^1_{r_1}+\cdots+\alpha^j_{r_j}\big).\tag{4.4.32}$$

Recall now (4.3.3) and notice that by (4.4.27) one has that $[E^{\pm}_{(l)}(\alpha_*,z),\beta'(\sigma;z)]=0$ for any $\beta\in\mathfrak{h}$. It then follows from (4.3.4)–(4.3.6) and (4.4.31)–(4.4.32) that for any $r_1,\dots,r_j\in\{1,2,\dots,l\}$,

$$\begin{aligned}&Y^{\sigma,(l)}_*\big(\iota(e_{r_1,\dots,r_j}),z\big)\\&\quad=C_3(\alpha)z^{C_2(\alpha)}E^-_{(l)}(\alpha_*,z)Y^{\sigma,(l)}\big(\iota(e_{r_1,\dots,r_j}),z\big)E^+_{(l)}(\alpha_*,z).\end{aligned}\tag{4.4.33}$$

Furthermore, since $v_{r_1,\dots,r_j}\in S((\tilde{\mathfrak{h}}^{\perp}_*)'_-)$, we get from (4.4.33) and (4.3.7) that

$$W_*(u_{r_1,\dots,r_j},z)=C_3(\alpha)z^{C_2(\alpha)}E^-_{(l)}(\alpha_*,z)W(u_{r_1,\dots,r_j},z)E^+_{(l)}(\alpha_*,z),$$

where $W(\cdot,z)$ stands for the unrelativized version of the operator $W_*(\cdot,z)$ defined in (4.3.7). Then (4.4.30) yields

$$\begin{aligned}&Y^{\sigma,(l)}_*(u_{r_1,\dots,r_j},z)\\&\quad=C_3(\alpha)z^{C_2(\alpha)}E^-_{(l)}(\alpha_*,z)Y^{\sigma,(l)}(u_{r_1,\dots,r_j},z)E^+_{(l)}(\alpha_*,z).\end{aligned}\tag{4.4.34}$$

From (4.4.29) and (4.4.34) we finally get

$$Y^{\sigma,(l)}_*(v,z)=C_3(\alpha)z^{C_2(\alpha)}E^-_{(l)}(\alpha_*,z)Y^{\sigma,(l)}(v,z)E^+_{(l)}(\alpha_*,z).\tag{4.4.35}$$

The lemma now follows from (4.4.35), Theorem 4.3.1, and Corollary 4.3.3. □

Let $\alpha\in\Phi$ and consider the operator on V_L^T determined by

$$Z(\alpha,z)=E^-_{(l)}(\alpha,z)x_\alpha(\sigma;z)zE^+_{(l)}(\alpha,z)\in\overline{U\big(\hat{\mathfrak{g}}[\sigma]\big)}\,[\![z^{1/6},z^{-1/6}]\!],\tag{4.4.36}$$

where $E^{\pm}_{(l)}(\cdot,z)$ are defined as in (4.4.27) and $\overline{U(\hat{\mathfrak{g}}[\sigma])}$ is a certain completion of $U(\hat{\mathfrak{g}}[\sigma])$ (see [4]). Recall the subalgebra $\overline{\mathcal{Y}}_{V_L^T}$ of $\operatorname{End}V_L^T$ from (4.4.24) and notice that $Z(\alpha,z)$ becomes exactly the Z-operator $Z(\alpha,\zeta)$ defined in [3, (3.18)] if one substitutes z by ζ^{-6} in (4.4.36). Then we can prove the following proposition.

Proposition 4.4.6. *$\overline{\mathcal{Y}}_{V_L^T}$ acts irreducibly on each of the spaces $\Omega_{k_0,l}$, $k_0\in\{0,1,\dots,k\}$.*

Proof. Let $k_0 \in \{0, 1, \ldots, k\}$. Note first that (4.3.11) and (4.4.23) imply in particular that the component operators of $\overline{Y}_*^{\sigma,(l)}(v, z)$, $v \in \Omega^A_{L(l,0)}$, preserve the vacuum space Ω_*^σ defined in (4.4.1). By Lemma 4.4.5 and (4.4.23), these operators also preserve the $\hat{\mathfrak{g}}[\sigma]$-module $L_{k_0,l}(\hat{\mathfrak{g}}[\sigma])$, and then it follows from (4.4.26) that $\overline{\mathcal{Y}}_{V_L^T}$ preserves $\Omega_{k_0,l}$.

In order to prove that the action of $\overline{\mathcal{Y}}_{V_L^T}$ on $\Omega_{k_0,l}$ is irreducible, we note that the operators $\overline{Y}_*^{\sigma,(l)}(i(x_\alpha), z)$, $\alpha \in \Phi$, are in fact closely related to the operators $Z(\alpha, z)$ defined in (4.4.36). Indeed, a simplified version of the computations in Lemma 4.4.5 (in this case just a straightforward consequence of (4.3.4)–(4.3.6), (4.4.8), Theorem 4.3.1, Remark 4.3.2, and the fact that $\tau(\alpha_i)\tau(\alpha_i')^{-1} = \tau(\alpha_*/l)$ for $i = 1, \ldots, l$) yields

$$\begin{aligned}
\pi^\sigma Z(\alpha, z) &= E^-_{(l)}(\alpha_*, z) Y^{\sigma,(l)}\big(i(x_\alpha), z\big) z E^+_{(l)}(\alpha_*, z) \\
&= \sum_{i=1}^{l} E^-_{(l)}(\alpha_*, z) Y^{\sigma,(1)}\big(\iota(e_{\alpha_i}), z\big) z E^+_{(l)}(\alpha_*, z) \\
&= (6z)^{-1} z \sum_{i=1}^{l} \tau(\alpha_i) E^-_{(1)}(-\alpha_i + \alpha_*/l, z) E^+_{(1)}(-\alpha_i + \alpha_*/l, z) \\
&= (6z)^{1/l} z \sum_{i=1}^{l} \tau(\alpha_i) E^-_{*,(1)}(-\alpha_i, z) E^+_{*,(1)}(-\alpha_i, z)(6z)^{-1+1/l} \\
&= \tau\left(\frac{\alpha_*}{l}\right) 6^{-1/l} z^{1-1/l} \sum_{i=1}^{l} Y_*^{\sigma,(1)}\big(\iota(e_{\alpha_i}), z\big) \\
&= \tau\left(\frac{\alpha_*}{l}\right) 6^{-1/l} z^{1-1/l} Y_*^{\sigma,(l)}\big(i(x_\alpha), z\big),
\end{aligned} \tag{4.4.37}$$

so that the operator $\pi^\sigma Z(\alpha, z)$ is essentially a relative σ-twisted vertex operator. Denote by $\mathcal{Z}'_{V_L^T}$ the subalgebra of $\operatorname{End} V_L^T$ generated by the component operators of the fields $\pi^\sigma Z(\alpha, z)$, $\alpha \in \Phi$. Clearly, $\mathcal{Z}'_{V_L^T} \subset \overline{\mathcal{Y}}_{V_L^T}$. It then follows from [3, Proposition 3.1] and the "equivalence theorem" [3, Theorems 5.5 and 5.6] that $\mathcal{Z}'_{V_L^T}$ acts irreducibly on $\Omega_{k_0,l}$ so that the same must be true for $\overline{\mathcal{Y}}_{V_L^T}$. $\square$

Recall from (3.2.24)–(3.2.25) and (4.4.17) that the Virasoro element $\omega_2 \in \Omega_l^0$ induces a $\frac{1}{6}\mathbf{Z}$-gradation on $\Omega_{k_0,l}$ via the operator $L^2(0) = \operatorname{Res}_z z Y^\sigma(\omega_2, z)$, and that the corresponding q-trace is given by $f_{k_0,l}$ (cf. (3.2.26) and (3.2.30)).

Lemma 4.4.7. *With the above notations, one has*

$$\overline{Y}_*^{\sigma,(l)}(\omega_2 + W_{L(l,0)}, z) = Y^\sigma(\omega_2, z). \tag{4.4.38}$$

In particular, the q-trace

$$\chi^{\sigma}_{k_0,k}(q) := \operatorname{tr}_{\Omega_{k_0,l}} q^{\overline{L}_*(0)-c_2(k)/24},$$
$$\text{where}\quad \overline{L}_*(0) = \operatorname{Res}_z z\overline{Y}^{\sigma,(l)}_*(\omega_2 + W_{L(l,0)}, z),$$

is well defined and coincides with $f_{k_0,l}(q)$.

Proof. One can use the definition of relative σ-twisted vertex operators in Section 4.3 and imitate the proof of Lemma 4.4.2 in order to obtain from Corollary 4.3.3 that when acting on Ω^{σ}_*,

$$Y^{\sigma,(l)}_*(v,z) = Y^{\sigma,(l)}(v,z) = Y^{\sigma}(v,z) \quad \text{for } v \in \Omega^0_l. \tag{4.4.39}$$

By Proposition 4.4.6, Eq. (4.4.39) holds even as an identity between (series of) operators on $\Omega_{k_0,l}$. Then (4.4.38) follows from Lemma 4.4.1, (4.4.21), (4.4.23) and (4.4.39). □

Summarizing, we have following theorem.

Theorem 4.4.8. *Let* $k_0 \in \{0,1,\ldots,k\}$ *and* $\Omega_{k_0,l} \in \mathcal{O}(k)$. *Then the structure*

$$\left(\Omega^A_{L(l,0)}, \overline{Y}^{(l)}_*, \iota(A), \omega_2 \otimes \iota(A), l, G, (\cdot,\cdot)\right)$$

is a simple GVOA of rank $c_2(k)$ *which acts irreducibly on* $\Omega_{k_0,l}$ *by means of the operators* $\overline{Y}^{\sigma,(l)}_*(\cdot,z)$, *in such a way that* $\Omega_{k_0,l}$ *has a well-defined* q*-trace* $\chi^{\sigma}_{k_0,k}(q)$ *given by*

$$\chi^{\sigma}_{k_0,k}(q) = f_{k_0,l}(q) = \chi^{\tau}_{k_0,k}(q),$$

where $\chi^{\tau}_{k_0,k}(q)$ *as in Theorem* 3.2.7.

Remark 4.4.9. It follows from Lemma 4.4.5 and Proposition 4.4.6 that the whole VOA $L(l,0)$ acts in fact irreducibly on each of the spaces $\Omega_{k_0,l}$, $k_0 \in \{0,1,\ldots,k\}$, by means of the operators $\overline{Y}^{\sigma,(l)}_*(\cdot,z)$. Furthermore, under this action the Virasoro element $\omega' \in L(l,0)$ (see (4.4.17) and (3.2.20)) induces the same q-trace $\chi^{\sigma}_{k_0,k}(q)$ for the space $\Omega_{k_0,l}$. The techniques of Lemma 4.4.5 and Proposition 4.4.6 seem therefore flexible enough to allow the construction of structures of (G)VOA type which are "larger" than $\Omega^A_{L(l,0)}$ and still satisfy the properties described in Theorem 4.4.8 (except possibly for the rank).

We conclude our partial discussion of Problems 1 and 2 with some remarks about potential further developments. A natural context for Theorem 4.4.8 would obviously be the "twisted representation theory" of the GVOA $\Omega^A_{L(l,0)}$. This fact alone motivates therefore an axiomatic study of the notion of "twisted module for a GVOA", of which the spaces $\Omega_{k_0,l}$ should be natural examples. It is quite clear that the techniques used for proving Lemma 4.4.5 and Proposition 4.4.6 are easily adapted to the more general setting of an arbitrary positive definite even lattice, thus extending to a reasonably large class of examples. We therefore believe that formulas such as (4.4.35) and (4.4.37) together with the duality properties of ordinary twisted modules could lead to a relatively simple-looking Jacobi identity for a twisted GVOA-module, which is usually the main axiom for structures of this type. (The very technical nature of the generalized twisted Jacobi identity established in [15] makes this identity somewhat inappropriate for such purposes.) The appropriate axiomatic setting once developed, one expects the GVOA $\Omega^A_{L(l,0)}$ to be "σ-rational" and the spaces $\Omega_{k_0,l}$, $k_0 \in \{0, 1, \dots, k\}$, to be in fact its only simple σ-twisted modules, in view of the "equivalence theorem" [3, Theorems 5.5 and 5.6]. Theorem 3.2.7, Theorem 4.4.8, and Remark 3.2.8(ii) would then suggest a strong connection between the τ-twisted representation theory of the τ-rational VOA $V_k(A_1^{(1)})^{\otimes 2}$ and the "σ-twisted representation theory" of the "σ-rational" GVOA $\Omega^A_{L(l,0)}$, or between appropriate substructures of these algebras. We believe that such connections would eventually answer Problems 1 and 2 in the affirmative.

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Appendix A. Some character formulas

We give here the formula for the unspecialized character of $V = L(3\Lambda_0; A_2^{(1)})$. Recall the notations from Section 2.1 in this case. In particular, α_0, α_1, α_2 are the simple roots and $\delta = \alpha_0 + \alpha_1 + \alpha_2$. Set $u = \mathrm{e}^{-\alpha_0}$, $v = \mathrm{e}^{-\alpha_1}$, $w = \mathrm{e}^{\alpha_2}$. The computations were made by using the techniques developed in [8,38]; that is, by expressing $\operatorname{ch} V$ in terms of the string functions for the maximal dominant weights and explicitly calculating their orbits under the affine Weyl group. One gets

$$\begin{aligned}
\mathrm{e}^{-3\Lambda_0} \operatorname{ch} V = {} & \left(\sum_{k=0}^{\infty} \dim V_{3\Lambda_0 - k\delta} (uvw)^k \right) \\
& \times \left[\sum_{m,n \in \mathbf{Z}} (uvw)^{3(m^2+n^2-mn)} v^{-3m} w^{-3n} \right] \\
& + \left(\sum_{k=0}^{\infty} \dim V_{3\Lambda_0 - \alpha_0 - k\delta} (uvw)^{k+1} \right) \\
& \times \left[\sum_{m,n \in \mathbf{Z}} (uvw)^{3(m^2+n^2-mn)-(2m-n)} v^{1-3m} w^{-3n} \right. \\
& \qquad \times \left(1 + (uvw)^{2(2m-n)} v^{-2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m,n\in\mathbf{Z}}(uvw)^{3(m^2+n^2-mn)-(2n-m)}v^{-3m}w^{1-3n}\\
&\qquad\times\bigl(1+(uvw)^{2(2n-m)}w^{-2}\bigr)\\
&+\sum_{m,n\in\mathbf{Z}}(uvw)^{3(m^2+n^2-mn)-(m+n)}v^{1-3m}w^{1-3n}\\
&\qquad\times\bigl(1+(uvw)^{2(m+n)}v^{-2}w^{-2}\bigr)\Bigr]\\
&+\left(\sum_{k=0}^{\infty}\dim V_{3\Lambda_0-2\alpha_0-\alpha_2-k\delta}(uvw)^k\right)\\
&\times\left[\sum_{m,n\in\mathbf{Z}}(uvw)^{3(m^2+n^2-mn-m-n)+2}v^{1+3(2n-m)}w^{-1+3n}\right]\\
&+\left(\sum_{k=0}^{\infty}\dim V_{3\Lambda_0-2\alpha_0-\alpha_1-k\delta}(uvw)^k\right)\\
&\times\left[\sum_{m,n\in\mathbf{Z}}(uvw)^{3(m^2+n^2-mn+m-n)+2}v^{-1-3m}w^{1-3n}\right]. \qquad (\mathrm{A}.1)
\end{aligned}
$$

Using the table Affine A_2-Level 3-Class 0-Rp 4 from [39, p. 408], one can obtain concrete expressions for every specialization $F_{(s_0,s_1,s_2)}(\mathrm{e}^{-3\Lambda_0}\operatorname{ch}V)$ “near the top” of the module V. For instance,

$$
\begin{aligned}
&F_{(4,1,1)}\bigl(\mathrm{e}^{-3\Lambda_0}\operatorname{ch}V\bigr)\big|_{q\to q^{1/6}}\\
&\quad=1+q^{4/6}+2q^{5/6}+2q+2q^{7/6}+\cdots+46q^{20/6}+\mathcal{O}\bigl(q^{21/6}\bigr). \qquad (\mathrm{A}.2)
\end{aligned}
$$

Unfortunately, as shown by computer experiments, it is not likely that (A.2) could be rewritten as a product formula in which the graded dimension of a symmetric algebra is easily identifiable (cf. Remark 3.2.6). The same can actually be said about the character

$$F_{(1,0,0)}\bigl(\mathrm{e}^{-3\Lambda_0}\operatorname{ch}V\bigr)=1+8q+44q^2+192q^3+\cdots+\mathcal{O}\bigl(q^{23}\bigr),$$

which up to the factor $q^{-1/6}$ is just the q-trace of the VOA $V_3(A_2^{(1)})$. The latter formula may of course be used for computing the graded dimension of the vacuum space Ω_3. Indeed, after dividing out the graded dimension of the Fock space $M(3)$ (which by (3.2.18) is $\prod_{n=1}^{\infty}(1-q^n)^{-2}$), one gets

$$\dim_* \Omega_3=1+6q+27q^2+98q^3+\cdots+\mathcal{O}\bigl(q^{23}\bigr).$$

For the purposes described in Section 3.2, we are in fact more interested in comparing the q-traces of the VOAs Ω_3^0 and $V_1(A_1^{(1)})^{\otimes 2}$. If we substitute u by $qv^{-1}w^{-1}$ in formula (A.1), then multiply it with $\prod_{n=1}^{\infty}(1-q^n)^2$ and collect the terms containing only zero powers of v and w, we get

$$\begin{aligned}&\operatorname{tr}_{\Omega_3^0} q^{L^2(0)-c_2(1)/24}\\&\quad= q^{-1/12}\bigl(1+3q^2+8q^3+16q^4+\cdots+\mathcal{O}\bigl(q^{22}\bigr)\bigr),\end{aligned}\tag{A.3}$$

since $c_2(1)=2$ (cf. Section 3.2). On the other hand, using the formulas of [39, Section 21.8] for the unspecialized character of $L(\Lambda_0; A_1^{(1)}) = V_1(A_1^{(1)})$, one gets (recall from Section 3.2 that $c_1(1)=1$)

$$\begin{aligned}&\operatorname{tr}_{V_1(A_1^{(1)})} q^{L(0)-c_1(1)/24}\\&\quad= q^{-1/24}\bigl(1+3q+4q^2+7q^3+\cdots+\mathcal{O}\bigl(q^{23}\bigr)\bigr),\end{aligned}\tag{A.4}$$

and, consequently,

$$\begin{aligned}&\operatorname{tr}_{V_1(A_1^{(1)})^{\otimes 2}} q^{L_{\otimes}(0)-c_2(1)/24}\\&\quad= q^{-1/12}\bigl(1+6q+17q^2+38q^3+\cdots+\mathcal{O}\bigl(q^{23}\bigr)\bigr).\end{aligned}\tag{A.5}$$

As we already noted in Section 3.2, the conformal vectors of Ω_3^0 and $V_1(A_1^{(1)})$ are such that $\operatorname{rank}\Omega_3^0 = 2\operatorname{rank}V_1(A_1^{(1)})$, which obviously prevents these spaces from being isomorphic as VOAs. Formulas (A.3) and (A.4) show actually that Ω_3^0 and $V_1(A_1^{(1)})$ are not even isomorphic as graded vector spaces. Moreover, (A.3) and (A.5) imply that although they have the same rank, the VOAs Ω_3^0 and $V_1(A_1^{(1)})^{\otimes 2}$ cannot be isomorphic either.

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